Transfer Matrices for the Partition Function of the Potts Model on Cyclic and Möbius Lattice Strips

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We present a method for calculating transfer matrices for the q-state Potts model partition functions $Z(G,q,v)$, for arbitrary q and temperature variable v, on cyclic and Möbius strip graphs G of the square (sq), triangular (tri), and honeycomb (hc) lattices of width L_y vertices and of arbitrarily great length L_x vertices. For the cyclic case we express the partition function as $Z(\Lambda, L_y \times L_x, q, v) = \sum_{d=0}^{L_y} c^{(d)} Tr[(T_{Z,\Lambda,L_y,d})^m]$, where Λ denotes lattice type, $c^{(d)}$ are specified polynomials of degree d in q, $T_{Z,\Lambda,L_y,d}$ is the transfer matrix in the degree-d subspace, and $m = L_x$ $(L_x/2)$ for $\Lambda = sq,\ tri$ (hc), respectively. An analogous formula is given for Möbius strips. We exhibit a method for calculating $T_{Z,\Lambda,L_y,d}$ for arbitrary L_y. Explicit results for arbitrary L_y are given for $T_{Z,\Lambda,L_y,d}$ with $d = L_y$ and $d = L_y - 1$. In particular, we find very simple formulas the determinant $det(T_{Z,\Lambda,L_y,d})$, and trace $Tr(T_{Z,\Lambda,L_y})$. Corresponding results are given for the equivalent Tutte polynomials for these lattice strips and illustrative examples are included. We also present formulas for self-dual cyclic strips of the square lattice.

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I. INTRODUCTION

The q-state Potts model has served as a valuable model for the study of phase transitions and critical phenomena [1,2]. On a lattice, or, more generally, on a (connected) graph G, at temperature T, this model is defined by the partition function

$$Z(G,q,v) = \sum_{\{\sigma_n\}} e^{-\beta\mathcal{H}} \tag{1.1}$$

with the (zero-field) Hamiltonian

$$\mathcal{H} = -J\sum_{\langle ij\rangle} \delta_{\sigma_i\sigma_j} \tag{1.2}$$

where $\sigma_i = 1, ..., q$ are the spin variables on each vertex (site) $i \in G$; $\beta = (k_BT)^{-1}$; and $\langle ij\rangle$ denotes pairs of adjacent vertices. The graph $G = G(V,E)$ is defined by its vertex set V and its edge set E; we denote the number of vertices of G as $n = n(G) = |V|$ and the number of edges of G as $e(G) = |E|$. We use the notation

$$K = \beta J \ , \quad v = e^K - 1 \tag{1.3}$$

so that the physical ranges are $v \ge 0$ for the Potts ferromagnet, and $-1 \le v \le 0$ for Potts antiferromagnet, corresponding to $0 \le T \le \infty$. One defines the (reduced) free energy per site $f = -\beta F$, where F is the actual free energy, via $f(\{G\},q,v) = \lim_{n\to\infty} \ln[Z(G,q,v)^{1/n}]$, where we use the symbol $\{G\}$ to denote the formal limit $\lim_{n\to\infty} G$ for a given family of graphs. In the present context, this $n \to \infty$ limit corresponds to the limit of infinite length for a strip graph of the square lattice of fixed width and some prescribed boundary conditions.

In this paper we shall present transfer matrices for the q-state Potts model partition functions $Z(G,q,v)$, for arbitrary q and temperature variable v, on cyclic and Möbius strip graphs G of the square, triangular, and honeycomb lattices of width L_y vertices and of arbitrarily great length L_x vertices. We label the lattice type as Λ and abbreviate the three respective types as *sq*, *tri*, and *hc*. Each strip involves a longitudinal repetition of m copies of a particular subgraph. For the square-lattice strips, this is a column of squares. It is convenient to represent the strip of the triangular lattice as obtained from the corresponding strip of the square lattice with additional diagonal edges connecting, say, the upper-left to lower-right vertices in each square. In both these cases, the length is $L_x = m$ vertices. We represent the strip of the honeycomb lattice in the form of bricks oriented horizontally. In this case, since there are two vertices in 1-1 correspondence with each horizontal side of a brick, $L_x = 2m$ vertices. Summarizing for all of three lattices, the relation between the number of vertices and the number of repeated copies is

$$L_x = \begin{cases} m & \text{if } \Lambda = sq \quad \text{or} \quad tri \quad \text{or } G_D \\ 2m & \text{if } \Lambda = hc \end{cases} \tag{1.4}$$

Here G_D is the cyclic self-dual strip of the square lattice, to be discussed further below. For the cyclic case the partition function has the general form [11,12]

$$Z(\Lambda, L_y \times L_x, cyc., q, v) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(\Lambda,L_y,d)} (\lambda_{Z,\Lambda,L,d,j})^m \tag{1.5}$$

In terms of transfer matrices, this can be written as

$$Z(\Lambda, L_y \times L_x, cyc., q, v) = \sum_{d=0}^{L_y} c^{(d)} Tr[(T_{Z,\Lambda,L_y,d})^m] \tag{1.6}$$

where $c^{(d)}$ are polynomials of degree d in q defined below, and the transfer matrices in the degree-d subspace $T_{Z,\Lambda,L_y,d}$ and their eigenvalues $\lambda_{Z,\Lambda,L_y,d,j}$, $j = 1, ..., n_Z(\Lambda, L_y, d)$ are independent of the length of the strip length L_x. Here we shall construct an analogous formula for Möbius strips. We exhibit our method for calculating $T_{Z,\Lambda,L_y,d}$ for arbitrary L_y. Explicit results for arbitrary L_y are given for (i) $T_{Z,\Lambda,L_y,d}$ with $d = L_y$ and $d = L_y - 1$, (ii) the determinant $det(T_{Z,\Lambda,L_y,d})$, and (iii) the trace $Tr(T_{Z,\Lambda,L_y})$. Corresponding results are given for the equivalent Tutte polynomials for these lattice strips and illustrative examples are included. We have calculated the transfer matrices up to widths $L_y = 5$ for the square, triangular, and honeycomb lattices and $L_y = 4$ for the cyclic self-dual strip of the square lattice. Since the dimensions of these matrices increase rapidly with strip width (e.g., $dim(T_{Z,\Lambda,L_y,d}) = 14, 28, 20, 7$ for $L_y = 4$ and $0 \le d \le 3$), it is not feasible to present many of the explicit results here; instead, we concentrate on general methods and results that hold for arbitrary L_y.

Various special cases of the Potts model partition function $Z(G, q, v)$ are of interest. For example, if one considers the case of antiferromagnetic spin-spin coupling, $J < 0$ and takes the temperature to zero, so that $K = -\infty$ and $v = -1$, then

$$Z(G, q, -1) = P(G, q) \tag{1.7}$$

where $P(G, q)$ is the chromatic polynomial (in q) expressing the number of ways of coloring the vertices of the graph G with q colors such that no two adjacent vertices have the same color [8,9].

We recall some previous related work. The partition function $Z(G, q, v)$ for the Potts model was calculated for arbitrary q and v on strips of the lattice Λ with cyclic or Möbius longitudinal boundary conditions was calculated for (i) $\Lambda = sq$, $L_y = 2$ in [11] and the

nature of the coefficients $c^{(d)}$ and subspace dimensions $n_Z(\Lambda, L_y, d)$ given in [12], (ii) $\Lambda = sq$, $L_y = 3$ in [13], (iii) $\Lambda = tri$, $L_y = 3$ in [14], (iv) for $\Lambda = hc$ in [15], (v) for cyclic self-dual strips of the square lattice with $L_y = 1, 2, 3$ in [16–18], and (vi) for $L_y = 2, 3$ on the square-lattice with next-nearest-neighbor spin-spin couplings in [19,20]. Some general structural properties for lattice strips with arbitrary width were given in [12,17,21]. Matrix methods for calculating chromatic polynomials were developed and used in [22,23], [24,25] and more recently in [26]- [28]. Ref. [29] developed transfer matrix methods for both $Z(G, q, v)$ and the special case $v = -1$ of chromatic polynomials on strips of the square lattice with free longitudinal boundary conditions and used them to calculate the latter polynomials for a large variety of widths. These have been termed transfer matrices in the Fortuin-Kasteleyn representation (see eq. (1.8) below). These methods were applied to calculate the full Potts model partition function for strips of the square and triangular lattices with free boundary conditions and a number of widths in Refs. [30,31]. A number of calculations have been done for the special case of chromatic polynomials of lattice strip graphs; we do not review these here but refer the reader to the references in [34]. In a different direction, we mention calculations for arbitrary q and v on rectangular lattice patches for the purpose of studying distributions of zeros [32,33]; we also do not review these studies here since our present focus is calculations for strips of arbitrarily great length.

There are several motivations for presenting these transfer matrix methods for calculating Potts model partition functions on cyclic and Möbius lattice strips and general results that we have obtained for lattice strips of arbitrary width as well as length. Clearly, new exact results on the Potts model are of value in their own right. The transfer matrices are also a convenient way to calculate partition functions. In explaining this, one should note that a very compact way of expressing the relevant information is in the form of generating functions for $Z(G, q, v)$ or in the form where the eigenvalues are given as roots of the characteristic polynomials. For a particular degree-d subspace, it requires $n_Z(\Lambda, L_y, d)$ coefficients to specify these characteristic polynomials. In contrast, it requires $n_Z(\Lambda, L_y, d)^2$ entries (some of which may be the same) to specify the transfer matrix $T_{Z,\Lambda,L_y,d}$. However, although the transfer matrix is not the most compact way of presenting the necessary information to specify $Z(G, q, v)$, it is, nevertheless, as convenient as the generating function method for calculating the partition function since one only needs to compute traces of powers of the matrices. Moreover, except for the lowest few values of the strip width L_y, the characteristic polynomials are often of fifth order or higher, so that it is not possible to obtain explicit algebraic solutions for the eigenvalues. In our previous work we have made use of the theorem on symmetric polynomial functions of roots of an algebraic equation [36,37], which states that such functions can be expressed (via Newton identities) in terms of the

coefficients of the algebraic equation. The particular type of symmetric polynomial function of the roots that is relevant for the Potts model partition functions on cyclic strips is the sum of m'th powers of these roots. Although the theorem on symmetric polynomial functions of roots of algebraic equations guarantees that these sums of m'th powers are expressible in terms of the coefficients of the equations, which are polynomials in q and v, it does not imply that they are particularly simple, and, indeed, the Potts model partition functions of moderately long lattice strips are quite lengthy expressions. Since the determinant and trace of a matrix are, respectively, the product and sum of the eigenvalues, they are both symmetric polynomial function of these eigenvalues (roots of the characteristic polynomial). The well-known expressions for the determinant and trace of a matrix in terms of coefficients of the characteristic polynomial are examples of the above-mentioned theorem. The very simple formulas that we have obtained for determinants and traces of transfer matrices provide further motivation for the present exposition.

Let $G' = (V, E')$ be a spanning subgraph of G, i.e. a subgraph having the same vertex set V and an edge set $E' \subseteq E$. $Z(G, q, v)$ can be written as the sum [3,4]

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{|E'|} \tag{1.8}$$

where $k(G')$ denotes the number of connected components of G'. Since we only consider connected graphs G, we have $k(G) = 1$. The formula (1.8) enables one to generalize q from $\mathbb{Z}_+$ to $\mathbb{R}_+$ for physical ferromagnetic v. More generally, eq. (1.8) allows one to generalize both q and v to complex values, as is necessary when studying zeros of the partition function in the complex q and v planes.

The Potts model partition function $Z(G, q, v)$ is equivalent to an object of considerable current interest in mathematical graph theory, the Tutte polynomial, $T(G, x, y)$, given by [5]- [7]

$$T(G, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)} (y-1)^{c(G')} \tag{1.9}$$

where $c(G') = |E'| + k(G') - |V|$ is the number of independent circuits in G'. Now let

$$x = 1 + \frac{q}{v}, \quad y = v + 1 \tag{1.10}$$

so that

$$q = (x-1)(y-1) \ . \tag{1.11}$$

Then the equivalence between the Potts model partition function and the Tutte polynomial for a graph G is

$$Z(G,q,v) = (x-1)^{k(G)}(y-1)^{n(G)}T(G,x,y) \ . \tag{1.12}$$

Given this equivalence, we can express results either in Potts or Tutte form. We will use both, since each has its own particular advantages. The Potts model form, involving the variables q and v is convenient for physical applications, since q specifies the number of states and determines the universality class of the transition, and v is the temperature variable. The Tutte form has the advantage that many expressions are simpler when written in terms of the Tutte variables x and y.

We recall that for a planar graph $G = (V,E)$, one defines the (planar) dual graph G^* as the graph obtained by replacing each vertex (face) of G by a face (vertex) of G^* and connecting the vertices of the resultant G^* by edges. The graph G is self-dual if and only if $G = G^*$. For a planar graph G_{pl}, it is evident from the definition (1.9) that the Tutte polynomial satisfies

$$T(G_{pl},x,y) = T(G^*_{pl},y,x) \ . \tag{1.13}$$

Equivalently,

$$Z(G_{pl},q,v) = v^{e(G_{pl})}q^{-c(G_{pl})}Z(G^*_{pl},q,\frac{q}{v}) \ . \tag{1.14}$$

The coefficients in eq. (1.5) are

$$c^{(d)} = U_{2d}(q^{1/2}/2) = \sum_{j=0}^{d}(-1)^j \binom{2d-j}{j} q^{d-j} \tag{1.15}$$

with $U_n(x)$ being the Chebyshev polynomial of the second kind. The first few of these coefficients are $c^{(0)} = 1$, $c^{(1)} = q-1$, $c^{(2)} = q^2-3q+1$, and $c^{(3)} = q^3-5q^2+6q-1$. The $c^{(d)}$'s play a role analogous to multiplicities of eigenvalues $\lambda_{Z,\Lambda,L_y,d,j}$, although this identification is formal, since $c^{(d)}$ may be zero or negative for the physical values $q = 1,2,3$ [12]. For $q \geq 4$, $c^{(d)}$ is positive-definite. From (1.12), one can write the Tutte polynomial as

$$T(\Lambda, L_y \times L_x, cyc., x, y) = \frac{1}{x-1}\sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(\Lambda,L_y,d)} (\lambda_{T,\Lambda,L_y,d,j})^m \tag{1.16}$$

where m is given in terms of L_x by eq. (1.4) and it is convenient to factor out a factor of $1/(x-1)$. (This factor is always cancelled, since the Tutte polynomial is a polynomial in x as well as y.) In terms of transfer matrices,

$$T(\Lambda, L_y \times L_x, cyc., x, y) = \frac{1}{x-1}\sum_{d=0}^{L_y} c^{(d)} Tr[(T_{T,\Lambda,L_y,d})^m] \ . \tag{1.17}$$

The equivalence of eq. (1.17) and (1.6) to (1.16) and (1.5), respectively, relies upon the theorem that an arbitrary square matrix T can be put into (upper or lower) triangular form (tf) by a (unitary) similarity transformation [35]

$$UTU^{-1} = T_{tf} \ . \tag{1.18}$$

For definiteness, we consider the upper triangular form, for which $(T_{tf})_{ij} = 0$ if $i > j$. A matrix in triangular form has its eigenvalues λ_j as its diagonal elements and has the properties that (i) If S and T are upper triangular matrices with eigenvalues $\lambda_{S,j}$ and $\lambda_{T,j}$, then ST is also a triangular matrix, with diagonal elements $(ST)_{jj} = \lambda_{S,j}\lambda_{T,j}$. A corollary is that for an arbitrary $N \times N$ matrix T

$$Tr(T^m) = Tr[(T_{tf})^m] = \sum_{j=1}^{N} \lambda_j^m \ . \tag{1.19}$$

The dimension of $T_{Z,\Lambda,L_y,d}$, or equivalently, $T_{T,\Lambda,L_y,d}$, is [12]

$$n_Z(\Lambda, L_y, d) = n_T(\Lambda, L_y, d) = \frac{(2d+1)}{(L_y+d+1)} \binom{2L_y}{L_y - d} \quad \text{for} \quad \Lambda = sq, tri, hc \ . \tag{1.20}$$

for $0 \le d \le L_y$ and zero otherwise. The property that the numbers $n_Z(\Lambda, L_y, d)$ are the same for all three lattices $\Lambda = sq, tri, hc$, as was shown for $\Lambda = sq, tri$ in Ref. [12] and for $\Lambda = hc$ in [15]. For this reason we shall henceforth usually revert to our previous notation in Ref. [12], setting

$$n_Z(\Lambda, L_y, d) \equiv n_Z(L_y, d) \ , \quad \Lambda = sq, tri, hc \tag{1.21}$$

The formal quantity $n_Z(0, d)$ will appear in some determinant formulas below and is given by eq. (1.20) as the Kronecker delta, $n_Z(0, d) = \delta_{d,0}$. (Below we shall consider self-dual strips G_D of the square lattice, which have different dimensions $n_Z(G_D, L_y, d)$; for this case we shall include the G_D dependence in the notation.) Special cases of $n_Z(L_y, d)$ that are of interest here include

$$n_Z(L_y, L_y) = 1 \tag{1.22}$$

$$n_Z(L_y, L_y - 1) = 2L_y - 1 \tag{1.23}$$

$$n_Z(L_y, 0) = C_{L_y} \tag{1.24}$$

where here C_n is the Catalan number which occurs in combinatorics and is defined by

$$C_n = \frac{1}{(n+1)} \binom{2n}{n} \ . \tag{1.25}$$

(No confusion should result from our use of the same symbol C_n to denote the circuit graph with n vertices since the meaning will be clear from context). The first few Catalan numbers are $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, and $C_5 = 42$.

The full transfer matrix T_{X,Λ,L_y}, $X = Z, T$, has a block structure formally specified by

$$T_{X,\Lambda,L_y} = \bigoplus_{d=0}^{L_y} \prod T_{X,\Lambda,L_y,d} \quad X = Z, T \tag{1.26}$$

where the product $\prod T_{X,\Lambda,L_y,d}$ means a set of square blocks, each of dimension, $c^{(d)}$, of the form $\lambda_{X,\Lambda,L_y,d,j}$ times the identity matrix. The dimension of the total transfer matrix, i.e., the total number of eigenvalues $\lambda_{X,\Lambda,L_y,d,j}$, counting multiplicities, is thus

$$dim(T_{X,\Lambda,L_y}) = \sum_{d=0}^{L_y} dim(T_{X,\Lambda,L_y,d}) = \sum_{d=0}^{L_y} c^{(d)} n_Z(\Lambda, L_y, d) \quad X = Z, T \ . \tag{1.27}$$

As in our earlier work, we define $N_{Z,\Lambda,L_y,\lambda}$ as the total number of distinct eigenvalues of T_{X,Λ,L_y}, $X = Z, T$, i.e. the sum of the dimensions of the submatrices $T_{X,\Lambda,L_y,d}$, modulo the multiplicity $c^{(d)}$. This is [12]

$$N_{Z,\Lambda,L_y} = N_{T,\Lambda,L_y} = \sum_{d=0}^{L_y} n_Z(L_y, d) = \binom{2L_y}{L} \quad \text{for} \ \ \Lambda = sq, tri, hc \ . \tag{1.28}$$

II. TRANSFER MATRIX METHOD

There are several equivalent ways to calculate $Z(G, q, v)$ or $T(G, x, y)$ for these lattice strip graphs. One is to make iterative use of the deletion-contraction relations obeyed by $T(G, x, y)$. This yields a generating function whose denominator directly determines the $\lambda_{X,\Lambda,L_y,d,j}$'s where $X = Z, T$. This iterative method works also for strip graphs with free longitudinal boundary conditions. The transfer matrix method for lattice strips of the square and triangular lattices with free longitudinal boundary conditions (and free or cylindrical transverse boundary conditions) was explained in detail in Ref. [29]. For the free strip with width L_y, the bases of the transfer matrix are all of the possible non-crossing partitions of L_y vertices. (For the zero-temperature antiferromagnetic Potts model, the non-nearest-neighbor requirement is imposed.) The eigenvalues of the transfer matrix for a free strip are the same as the eigenvalues of the transfer matrix $T_{Z,\Lambda,L_y,d=0}$ in the degree $d = 0$ subspace (called "level

0" subspace in [28]) for the corresponding cyclic strip, and the dimension of this matrix is $n_Z(L_y, 0) = C_{L_y}$, the Catalan number, as in eq. (1.24). This was shown for the free strips of the square and triangular lattices in [29] and for the cyclic/Möbius strips in [12] and extended to the honeycomb lattice in [15]. As discussed in Refs. [26]- [28] for chromatic polynomials, which we generalize here to the full Potts model partition function, the degree $d = 1$ subspace is given by all of the possible non-crossing partitions with a color assignment to one vertex (with possible connections with other vertices), and the multiplicity is $q - 1 = c^{(1)}$. This follows because there are q possible ways of making this color assignment, but one of these has to be subtracted, since the effect of all the possible color assignments is equivalent to the choice of no specific color assignment, which has been taken into account in the level 0 subspace. In this derivation and subsequent ones we assume that q is an integer ≥ 4 to begin with, so that the multiplicities are positive-definite; we then analytically continue them downward to apply in the region $0 \leq q < 4$ where $c^{(d)}$ can be zero or negative. For the next subspace we consider all of the non-crossing partitions with two-color assignments to two separated vertices (with possible connections with other vertices). Now the multiplicity can be understood by the sieve formula of [26]- [28]. Since the two assigned colors should be different, there are $q(q-1)$ ways of making these assignments. This includes the q possible color assignments for each of the two vertices that have been considered in level 1 and hence these must be subtracted. In doing this, the no-color assignment was subtracted twice, and one of these has to be added back. Therefore, the multiplicity is

$$q(q-1) - 2q + 1 = q^2 - 3q + 1 = c^{(2)} . \tag{2.1}$$

By the same method, for the non-crossing partitions with the three-color assignment, the multiplicity is

$$q(q-1)^2 - 2q(q-1) - q^2 + 3q - 1 = q^3 - 5q^2 + 6q - 1 = c^{(3)} . \tag{2.2}$$

The multiplicity for the four-color assignment is

$$\begin{aligned} & q(q-1)^3 - 2q(q-1)^2 - 2q^2(q-1) + 3q(q-1) + 3q^2 - 4q + 1 \\ & = (q-1)(q^3 - 6q^2 + 9q - 1) = c^{(4)} . \end{aligned} \tag{2.3}$$

and so forth for higher values of d. We show the calculation for these multiplicities pictorially for $2 \leq d \leq 4$ in Fig. 1. In general, the multiplicity of the d-color assignment can be computed to be $c^{(d)}$ given in eq. (1.15). We list graphically all the possible partitions for $L_y = 2$ and $L_y = 3$ strips in Figs. 2 and 3, respectively, where white circles are the original

L_y vertices and each black circle corresponds to a specific color assignment. We remark that the connections between the black circles and white circles also obey the non-crossing restriction, so it is possible that two non-adjacent black circles represent the same color. In the following discussion, we will simply use the names white and black circles with the meaning understood. We denote the partitions $\mathcal{P}_{L_y,d}$ for $2 \le L_y \le 4$ as follows:

$$\mathcal{P}_{2,0} = \{I; 12\}\ , \qquad \mathcal{P}_{2,1} = \{\bar{2}; \bar{1}; \overline{12}\}\ , \qquad \mathcal{P}_{2,2} = \{\bar{1}, \bar{2}\} \tag{2.4}$$

$$\mathcal{P}_{3,0} = \{I; 12; 13; 23; 123\}\ , \qquad \mathcal{P}_{3,1} = \{\bar{3}; \bar{2}; \bar{1}; 12, \bar{3}; \overline{12}; \overline{13}; \overline{23}; 23, \bar{1}; \overline{123}\}\ ,$$

$$\mathcal{P}_{3,2} = \{\bar{2}, \bar{3}; \bar{1}, \bar{3}; \bar{1}, \bar{2}; \overline{12}, \bar{3}; \bar{1}, \overline{23}\}\ , \qquad \mathcal{P}_{3,3} = \{\bar{1}, \bar{2}, \bar{3}\} \tag{2.5}$$

$$\mathcal{P}_{4,0} = \{I; 12; 13; 14; 23; 24; 34; 12, 34; 14, 23; 123; 124; 134; 234; 1234\}\ ,$$

$$\mathcal{P}_{4,1} = \{\bar{4}; \bar{3}; \bar{2}; \bar{1}; 12, \bar{4}; 12, \bar{3}; \overline{12}; 13, \bar{4}; \overline{13}; \overline{14}; 23, \bar{4}; \overline{23}; 23, \bar{1}; \overline{24}; 24, \bar{1}; \overline{34}; 34, \bar{2}; 34, \bar{1}; 12, \overline{34};$$

$$34, \overline{12}; 23, \overline{14}; 123, \bar{4}; \overline{123}; \overline{124}; \overline{134}; \overline{234}; 234, \bar{1}; \overline{1234}\}\ ,$$

$$\mathcal{P}_{4,2} = \{\bar{3}, \bar{4}; \bar{2}, \bar{4}; \bar{1}, \bar{4}; \bar{2}, \bar{3}; \bar{1}, \bar{3}; \bar{1}, \bar{2}; 12, \bar{3}, \bar{4}; \overline{12}, \bar{4}; \overline{12}, \bar{3}; \overline{13}, \bar{4}; \overline{23}, \bar{4}; 23, \bar{1}, \bar{4}; \bar{1}, \overline{23}; \bar{1}, \overline{24}; \bar{2}, \overline{34};$$

$$\bar{1}, \overline{34}; 34, \bar{1}, \bar{2}; \overline{12}, \overline{34}; \overline{123}, \bar{4}; \bar{1}, \overline{234}\}\ ,$$

$$\mathcal{P}_{4,3} = \{\bar{2}, \bar{3}, \bar{4}; \bar{1}, \bar{3}, \bar{4}; \bar{1}, \bar{2}, \bar{4}; \bar{1}, \bar{2}, \bar{3}; \overline{12}, \bar{3}, \bar{4}; \bar{1}, \overline{23}, \bar{4}; \bar{1}, \bar{2}, \overline{34}\}\ ,$$

$$\mathcal{P}_{4,4} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\} \tag{2.6}$$

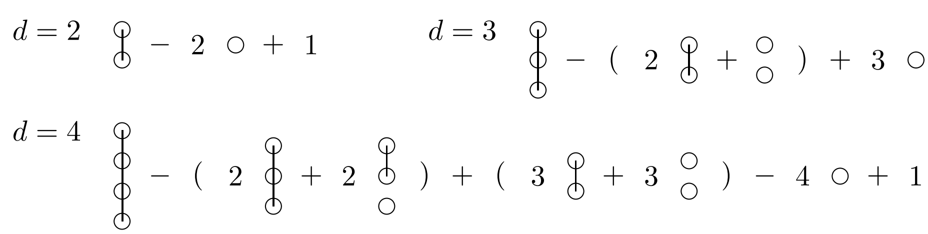

FIG. 1. Multiplicities of the transfer matrices for $2 \le d \le 4$.

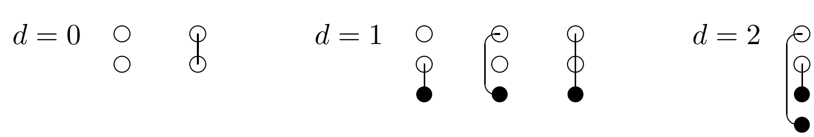

FIG. 2. Partitions for the $L_y = 2$ strip.

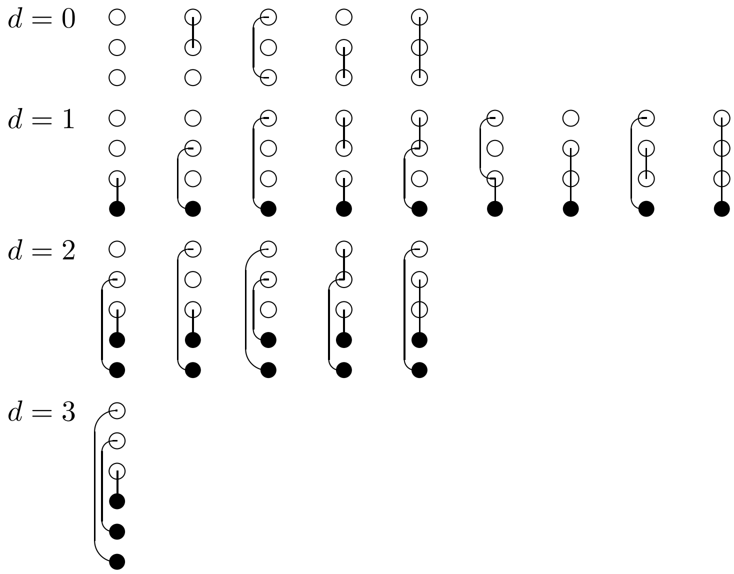

FIG. 3. Partitions for the $L_y = 3$ strip.

Using these methods, we can obtain $n_Z(L_y, d)$ given in Theorem 4 of [12] as follows. Since the maximum number of colors to assign is L_y for a strip with width L_y, it follows that $n_Z(L_y, d) = 0$ for $d > L_y$ and $n_Z(L_y, L_y) = 1$. It is elementary that $n_Z(1, 0) = 1$. The $d = 0$ partitions of a width-$(L_y + 1)$ strip can be obtained by either adding a unconnected white circle to the bottom of the $d = 0$ partitions of a width-L_y strip or converting the black circle of the $d = 1$ partitions of a width-L_y strip into a white circle. That is, $n_Z(L_y + 1, 0) = n_Z(L_y, 0) + n_Z(L_y, 1)$. Finally, the partitions of a width-$(L_y + 1)$ strip for $1 \le d \le L_y + 1$ can be obtained in one of the following four ways: (a) adding a pair of connected circles, one black and one white, (but not connected to any other vertex) above the highest black circle of the $d - 1$ partitions of a width-L_y strip; (b) adding a unconnected white circle above the highest black circle of the d partitions of a width-L_y strip; (c) adding a white circle above the highest black circle of the d partitions of a width-L_y strip and connecting these two circles; (d) converting the highest black circle of the $d+1$ partitions of a width-L_y strip into a white circle. Now the lowest white circle of the d partitions of a width-$(L_y + 1)$ strip can either connect to a black circle with or without ((c) or (a)) other connections to other white circles, or it does not connect to a black circle with or without ((d) or (b)) other connections to other white circles. Therefore, (a) to (d) exhaust all the possibilities, and we have $n_Z(L_y + 1, d) = n_Z(L_y, d - 1) + 2n_Z(L_y, d) + n_Z(L_y, d + 1)$ for $1 \le d \le L_y + 1$. This completes the proof. Indeed, these are precisely the equations that we obtained in a different manner in Ref. [12]. Solving these equations as we did in [12], it follows that the dimension of the transfer matrix $T_{Z,\Lambda,L_y,d}$ is the expression for $n_Z(L_y, d)$ given in eq. (1.20).

Similarly, for the zero-temperature antiferromagnetic Potts model, $n_P(\Lambda, L_y, d)$ for

$\Lambda = sq, tri$ given in Theorem 2 of [12] can be shown as follows. Now the adjacent white circles cannot connect to each other. Again, it is obvious that $n_P(L_y, d) = 0$ for $d > L_y$, $n_P(L_y, L_y) = 1$, and $n_P(1, 0) = 1$. The $d = 0$ partitions of a width-$(L_y + 1)$ strip can all be obtained by converting the black circle of the $d = 1$ partitions of a width-L_y strip into a white circle with the condition that the connection between the black circle and the lowest white circle, if there is one, must be deleted. That is, $n_P(L_y+1, 0) = n_P(L_y, 1)$. Finally, the partitions of a width-(L_y+1) strip for $1 \le d \le L+1$ can be obtained in one of the following four ways: (a) adding a pair of connected circles, one black and one white, (but not connected to any other vertex) above the highest black circle of the $d-1$ partitions of a width-L_y strip; (b) adding a unconnected white circle above the highest black circle of the d partitions of a width-L_y strip; (c) if the highest black circle of the $d+1$ partitions of a width-L_y strip connects to the lowest white circle, deleting this connection, converting this black circle into a white circle, and connecting it to the black circle below it; (d) if the highest black circle of the $d+1$ partitions of a width-L_y strip does not connect to the lowest white circle, converting it into a white circle. Now the lowest white circle of the d partitions of a width-(L_y+1) strip can either connect to a black circle with or without ((c) or (a)) other non-adjacent connections to other white circles, or it does not connect to a black circle with or without ((d) and (b)) other non-adjacent connections to other white circles. Therefore, (a) to (d) exhaust all the possibilities, and we have $n_P(L_y + 1, d) = n_P(L_y, d - 1) + n_P(L_y, d) + n_P(L_y, d + 1)$ for $1 \le d \le L + 1$. This completes the proof.

The construction of the transfer matrix for each level (= degree) d can be carried out by the methods similar to those for $d = 0$. For cyclic strips, the transfer matrix $T_{Z,\Lambda,L_y,d}$ is the product of the transverse and longitudinal parts, $H_{Z,\Lambda,L_y,d}$ and $V_{Z,\Lambda,L_y,d}$, which can be expressed as

$$
\begin{aligned}
H_{Z,sq,L_y,d} &= H_{Z,tri,L_y,d} = \prod_{i=1}^{L_y-1} (I + vJ_{L_y,d,i,i+1}) \\
H_{Z,hc,L_y,d,1} &= \prod_{i=1}^{[L_y/2]} (I + vJ_{L_y,d,2i-1,2i}) \ , \qquad H_{Z,hc,L_y,d,2} = \prod_{i=1}^{[(L_y-1)/2]} (I + vJ_{L_y,d,2i,2i+1}) \\
V_{Z,sq,L_y,d} &= V_{Z,hc,L_y,d} = \prod_{i=1}^{L_y} (vI + D_{L_y,d,i}) \\
V_{Z,tri,L_y,d} &= \prod_{i=1}^{L_y-1} [(vI + D_{L_y,d,i})(I + vJ_{L_y,d,i,i+1})](vI + D_{L_y,d,L_y}) \ ,
\end{aligned}
\tag{2.7}
$$

where $[\nu]$ denotes the integral part of ν, and I is the identity matrix of dimension $n_Z(L_y, d)$. $J_{L_y,d,i,i+1}$ is the join operator between vertices i and $i+1$, and $D_{L_y,d,i}$ is the detach operator

on vertex i [29]. We have

$$T_{Z,sq,L_y,d} = V_{Z,sq,L_y,d} H_{Z,sq,L_y,d} \ , \qquad T_{Z,tri,L_y,d} = V_{Z,tri,L_y,d} H_{Z,tri,L_y,d}$$

$$T_{Z,hc,L_y,d} = (V_{Z,hc,L_y,d} H_{Z,hc,L_y,d,2})(V_{Z,hc,L_y,d} H_{Z,hc,L_y,d,1}) \equiv T_{Z,hc,L_y,d,2} T_{Z,hc,L_y,d,1} \ . \tag{2.8}$$

Here we shall give results for this method for both cyclic and Möbius strips.

Consider two sets of L_y vertices and denote them as $i_1, i_2, ...i_{L_y}$ and $j_1, j_2, ...j_{L_y}$. For cyclic strips of the square lattice, the horizontal edges connecting these vertices are (i_1, j_1), (i_2, j_2), ..., (i_{L_y}, j_{L_y}). For Möbius strips, one set of horizontal edges becomes (i_1, j_{L_y}), (i_2, j_{L_y-1}), ..., (i_{L_y}, j_1). This corresponds to exchanging the pair of bases that switch to each other when the vertices reverse in order, i.e., the set of bases that do not have self-reflection symmetry with respect to the center of the L_y vertices. For example, among the partitions for the $L_y = 2$ strip in Fig. 2, the first partition $\bar{2}$ and the second partition $\bar{1}$ in $\mathcal{P}_{2,1}$ must be exchanged under this reflection. The pairs of partitions for the $L_y = 3$ strip in Fig. 3 are (1) the second partition 12 and the fourth partition 23 in $\mathcal{P}_{3,0}$, (2) the first partition $\bar{3}$ and the third partition $\bar{1}$ in $\mathcal{P}_{3,1}$, (3) the fifth partition $\overline{12}$ and the seventh partition $\overline{23}$ in $\mathcal{P}_{3,1}$, (4) the fourth partition $12, \bar{3}$ and the eighth partition $23, \bar{1}$ in $\mathcal{P}_{3,1}$, (5) the first partition $\bar{2}, \bar{3}$ and the third partition $\bar{1}, \bar{2}$ in $\mathcal{P}_{3,2}$, and (6) the fourth partition $\overline{12}, \bar{3}$ and the fifth partition $\bar{1}, \overline{23}$ in $\mathcal{P}_{3,2}$. For this specific set of edges of Möbius strips, the pairs of columns of $V_{Z,\Lambda,L_y,d}$ that correspond to these pairs of partitions should be exchanged, and these matrices will be denoted as $\tilde{V}_{Z,\Lambda,L_y,d}$. Equivalently, the same pairs of columns of $T_{Z,\Lambda,L_y,d}$ should be exchanged, and these matrices will be denoted as $\tilde{T}_{Z,\Lambda,L_y,d} = \tilde{V}_{Z,\Lambda,L_y,d} H_{Z,\Lambda,L_y,d}$ for $\Lambda = sq, tri$. There are two kinds of Möbius strips for the honeycomb lattice. When L_y is even, the number of vertices in the horizontal direction is even as for the cyclic strips, i.e., $L_x = 2m$. When L_y is odd, the number of vertices in the horizontal direction is odd, $L_x = 2m - 1$. Therefore, for the honeycomb lattice, we use the definition

$$\tilde{T}_{Z,hc,L_y,d} = \tilde{V}_{Z,hc,L_y,d} H_{Z,hc,L_y,d,1} \qquad \text{for odd } L_y$$

$$\tilde{T}_{Z,hc,L_y,d} = \tilde{V}_{Z,hc,L_y,d} H_{Z,hc,L_y,d,2} V_{Z,hc,L_y,d} H_{Z,hc,L_y,d,1} \qquad \text{for even } L_y \ . \tag{2.9}$$

As was discussed in [15] for the crossing-subgraph strips, the square of each eigenvalue of $\tilde{T}_{Z,hc,L_y,d}$ for odd L_y is an eigenvalue of $T_{Z,hc,L_y,d}$.

We have found in [12] the changes of coefficients for the square lattice when the longitudinal boundary condition is changed from cyclic to Möbius. In the context of our present transfer matrix formalism, we can express these changes of coefficients for the square, triangular and honeycomb lattices as follows:

$$c^{(0)} \to c^{(0)} \tag{2.10}$$

$$c^{(2k)} \to -c^{(k-1)} \ , \qquad 1 \le k \le \left[\frac{L_y}{2}\right] \tag{2.11}$$

$$c^{(2k+1)} \to c^{(k+1)} \ , \qquad 0 \le k \le \left[\frac{L_y-1}{2}\right] \ . \tag{2.12}$$

The Potts model partition function for Möbius strips is given by

$$\begin{aligned} Z(\Lambda, L \times L_x, Mb, q, v) = {} & c^{(0)} Tr[(T_{Z,\Lambda,L_y,0})^{m-1}\tilde{T}_{Z,\Lambda,L_y,0}] \\ & + \sum_{d=0}^{[(L_y-1)/2]} c^{(d+1)} Tr[(T_{Z,\Lambda,L_y,2d+1})^{m-1}\tilde{T}_{Z,\Lambda,L_y,2d+1}] \\ & - \sum_{d=1}^{[L_y/2]} c^{(d-1)} Tr[(T_{Z,\Lambda,L_y,2d})^{m-1}\tilde{T}_{Z,\Lambda,L_y,2d}] \end{aligned} \tag{2.13}$$

The corresponding formula for the Tutte polynomial is

$$\begin{aligned} T(\Lambda, L \times L_x, Mb, x, y) = {} & \frac{1}{x-1}\Big(c^{(0} Tr[(T_{T,\Lambda,L_y,0})^{m-1}\tilde{T}_{T,\Lambda,L_y,0}] \\ & + \sum_{d=0}^{[(L_y-1)/2]} c^{(d+1)} Tr[(T_{T,\Lambda,L_y,2d+1})^{m-1}\tilde{T}_{T,\Lambda,L_y,2d+1}] \\ & - \sum_{d=1}^{[L_y/2]} c^{(d-1)} Tr[(T_{T,\Lambda,L_y,2d})^{m-1}\tilde{T}_{T,\Lambda,L_y,2d}]\Big) \ . \end{aligned} \tag{2.14}$$

For the square lattice or the honeycomb lattice with L_y even, the eigenvalues of $\tilde{T}_{Z,\Lambda,L_y,d}$ are the same as those of $T_{Z,\Lambda,L_y,d}$ except for possible changes of signs. The number of eigenvalues with sign changes is equal to the number of column-exchanges from $T_{Z,\Lambda,L_y,d}$ to $\tilde{T}_{Z,\Lambda,L_y,d}$. Denote the number of eigenvalues that are the same for $T_{Z,sq,L_y,d}$ and $\tilde{T}_{Z,sq,,L_y,d}$ as $n_Z(sq, L_y, d, +)$, and the number of eigenvalues with different signs as $n_Z(sq, L_y, d, -)$. It is clear that

$$n_Z(sq, L_y, d, +) + n_Z(sq, L_y, d, -) = n_Z(sq, L_y, d) \ . \tag{2.15}$$

Define

$$\Delta n_Z(sq, L_y, d) \equiv n_Z(sq, L_y, d, +) - n_Z(sq, L_y, d, -) \tag{2.16}$$

which gives the number of partitions which have self-reflection symmetry. For example, among the partitions for the $L_y = 2$ strip in Fig. 2, the partitions I and 12 in $\mathcal{P}_{2,0}$, the third partition $\overline{12}$ in $\mathcal{P}_{2,1}$, and the partition $\bar{1}, \bar{2}$ in $\mathcal{P}_{2,2}$ have self-reflection symmetry. Among the partitions for the $L_y = 3$ strip in Fig. 3, it includes the first partition I, the third partition 13 and the fifth partition 123 in $\mathcal{P}_{3,0}$, the second partition $\bar{2}$, the sixth partition $\overline{13}$ and the ninth partition $\overline{123}$ in $\mathcal{P}_{3,1}$, the second partition $\bar{1}, \bar{3}$ in $\mathcal{P}_{3,2}$, and the partition in $\mathcal{P}_{3,3}$. We list $\Delta n_Z(sq, L_y, d)$ for $1 \le L_y \le 10$ in Table I in the appendix. The total number of these partitions for each L_y will be denoted as $\Delta N_{Z,L_y} = 2^{L_y}$. The relations between $\Delta n_Z(sq, L_y, d)$ are

$$\Delta n_Z(sq, 2n, 0) = 2\Delta n_Z(sq, 2n-1, 0) \ , \qquad \text{for } 0 < n$$

$$\begin{aligned}
\Delta n_Z(sq, 2n, 2m-1) &= \Delta n_Z(sq, 2n, 2m) \\
&= \Delta n_Z(sq, 2n-1, 2m-1) + \Delta n_Z(sq, 2n-1, 2m) \ , \qquad \text{for } 1 \le m \le n \\
\Delta n_Z(sq, 2n+1, 2m) &= \Delta n_Z(sq, 2n+1, 2m+1) \\
&= \Delta n_Z(sq, 2n, 2m) + \Delta n_Z(sq, 2n, 2m+1) \ , \qquad \text{for } 0 \le m \le n \ .
\end{aligned} \tag{2.17}$$

We also list $n_Z(sq, L_y, d, +)$ and $n_Z(sq, L_y, d, -)$ for $2 \le L_y \le 10$ in Table II in the appendix. Notice that $n_Z(sq, L_y, 0, +)$ is the number of λ_{Z,sq,FF,L_y} proved in [30]. Recall the numbers of $\lambda_{Z,sq,L_y,j}$ for the Möbius strips of the square lattice with coefficients $\pm c^{(d)}$, defined as $n_{Z,Mb}(L_y, d, \pm) \equiv n_{Z,Mb}(sq, L_y, d, \pm)$, have been given in [12]. With the eqs. (2.10) to (2.12), the relations between $n_Z(sq, L_y, d, \pm)$ and $n_{Z,Mb}(sq, L_y, d, \pm)$ are

$$\begin{aligned}
n_{Z,Mb}(sq, L_y, 0, \pm) &= n_Z(sq, L_y, 0, \pm) + n_Z(sq, L_y, 2, \mp) \\
n_{Z,Mb}(sq, L_y, k, \pm) &= n_Z(sq, L_y, 2k-1, \pm) + n_Z(sq, L_y, 2k+2, \mp) \ , \\
&\text{for} \quad 1 \le k \le \Big[\frac{L_y+1}{2}\Big] \ .
\end{aligned} \tag{2.18}$$

For a strip graph G_s of a lattice Λ with given boundary conditions, following our earlier notation [11] we denote the sum of coefficients (generalized multiplicities) c_{G_s} of the $\lambda_{Z,G_s,j}$ as C_{Z,G_s}. This sum is independent of the length m of the strip. For a cyclic strip graph this is given by [12]

$$C_{Z,\Lambda,cyc.,L_y} = \sum_{d=0}^{L_y} c^{(d)} n_Z(L_y, d) = q^{L_y} \quad \text{for} \ \ \Lambda = sq, tri, hc \ . \tag{2.19}$$

For the Möbius strip of the square lattice or the honeycomb lattice with L_y even, the sign changes of the eigenvalues of $\tilde{T}_{Z,\Lambda,L_y,d}$ can be considered as the sign changes of the coefficients. For these cases, the sum of coefficients is given by Theorem 8 of Ref. [12]:

$$C_{Z,(sq,hc),L_y,Mb} \equiv \sum_{j=1}^{N_{Z,L_y,\lambda}} c_{Z,L_y,Mb,j} = \sum_{d=0}^{d_{max}} \Delta n_{Z,Mb}(L_y,d) c^{(d)} = \begin{cases} q^{L_y/2} & \text{for even } L_y \\ q^{(L_y+1)/2} & \text{for odd } L_y \end{cases} \tag{2.20}$$

where

$$d_{max} = \begin{cases} \frac{L_y}{2} & \text{for even } L_y \\ \frac{(L_y+1)}{2} & \text{for odd } L_y \end{cases} \tag{2.21}$$

In previous work we have given results for the determinants for various strip graphs G_s (e.g., [13])

$$\det T_Z(G_s) = \prod_{j=1}^{N_{Z,G_s,\lambda}} (\lambda_{Z,G_s,j})^{c_{Z,G_s,j}} \tag{2.22}$$

and

$$\det T_P(G_s) = \prod_{j=1}^{N_{P,G_s,\lambda}} (\lambda_{P,G_s,j})^{c_{P,G_s,j}} \ . \tag{2.23}$$

and we shall extend these results to arbitrary width here. In the present context, these can be written, for cyclic strips, as

$$det(T_{Z,\Lambda,L_y}) = \prod_{d=0}^{L_y} [det(T_{Z,\Lambda,L_y,d})]^{c^{(d)}} \tag{2.24}$$

$$det(T_{T,\Lambda,L_y}) = \prod_{d=0}^{L_y} [det(T_{T,\Lambda,L_y,d})]^{c^{(d)}} \ . \tag{2.25}$$

From eq. (1.12) it follows that [11,15]

$$\lambda_{Z,\Lambda,L_y,d,j} = v^{pL_y} \lambda_{T,\Lambda,L_y,d,j} \tag{2.26}$$

and

$$T_{Z,\Lambda,L_y,d} = v^{pL_y} T_{T,\Lambda,L_y,d} \tag{2.27}$$

where

$$p = \begin{cases} 1 & \text{if } \Lambda = sq \ \text{ or } \ tri \ \text{ or } G_D \\ 2 & \text{if } \Lambda = hc \end{cases} \tag{2.28}$$

Note that the factor of $(x-1)$ in eq. (1.12) cancels the factor $1/(x-1)$ in eq. (1.17).

III. PROPERTIES OF TRANSFER MATRICES AT SPECIAL VALUES OF PARAMETERS

In this section we derive some properties of the transfer matrices $T_{Z,\Lambda,L_y,d}$ at special values of q and v, and, correspondingly, $T_{T,\Lambda,L_y,d}$ at special values of x and y.

A. $v = 0$

From (1.1) or (1.8) it follows that for any graph G, the Potts model partition function $Z(G,q,v)$ satisfies

$$Z(G,q,0) = q^{n(G)} \ . \tag{3.1}$$

Since this holds for arbitrary values of q, in the context of the lattice strips considered here, it implies

$$(T_{Z,\Lambda,L_y,d})_{v=0} = 0 \quad \text{for} \quad 1 \le d \le L_y \tag{3.2}$$

i.e. these are zero matrices. Secondly, restricting to cyclic strips for simplicity, and using the basic results $n = L_y L_x = L_y m$ for $\Lambda = sq, tri$ and $n = 2L_y m$ for $\Lambda = hc$, eq. (3.1) implies that

$$Tr[(T_{Z,\Lambda,cyc.,L_y,q,v})^m]_{v=0} = \begin{cases} q^{L_y m} & \text{for} \quad \Lambda = sq, tri \\ q^{2L_y m} & \text{for} \quad \Lambda = hc \end{cases} \tag{3.3}$$

With our explicit calculations, we find that

$$(T_{Z,\Lambda,L_y,0})_{jk} = 0 \quad \text{for} \ \ v = 0, \quad \text{and} \quad j \ge 2 \tag{3.4}$$

i.e., all rows of these matrices except the first vanish, and

$$(T_{Z,\Lambda,L_y,0})_{11} = q^{pL_y} \quad \text{for} \quad v = 0 \tag{3.5}$$

where p was given in eq. (2.28). For this $v = 0$ case, since all of the rows except the first are zero, the elements $(T_{Z,\Lambda,L_y,0})_{1k}$ for $k \ge 2$ do not enter into $Tr[(T_{Z,\Lambda,L_y,0})^m]$, which just reduces to the m'th power of the (1,1) element:

$$Tr[(T_{Z,\Lambda,L_y,0})^m] = [(T_{Z,\Lambda,L_y,0})_{11}]^m \ . \tag{3.6}$$

Corresponding to this, all of the eigenvalues $\lambda_{Z,\Lambda,L_y,d,j}$ vanish except for one, which is equal to $(T_{Z,\Lambda,L_y,0})_{11}$ in eq. (3.5). As will be seen, this is reflected in the property that $det(T_{Z,\Lambda,L_y,d})$

has a nonzero power of v as a factor for $L_y \ge 2$ for all of the lattice-Λ strips considered here. The restriction $L_y \ge 2$ is made because the strips of the triangular and honeycomb lattice are well-defined without degenerating for $L_y \ge 2$ and, in the case of the square lattice, for the case $L_y = 1$, both of the transfer matrices reduce to scalars, $T_{Z,sq,1,0} = q + v$ and $T_{Z,sq,1,1} = v$, the former of which is, in general, nonzero at $v = 0$. Note that the condition $v = 0$ is equivalent to the Tutte variable condition $y = 1$.

B. $q = 0$

Another fundamental relation that follows, e.g., by setting $q = 0$ in eq. (1.8), is

$$Z(G, 0, v) = 0 \ . \tag{3.7}$$

Now the coefficients $c^{(d)}$ evaluated at $q = 0$ satisfy [12]

$$c^{(d)} = (-1)^d \quad \text{for} \quad q = 0 \ . \tag{3.8}$$

Hence, in terms of transfer matrices, we derive the sum rule

$$\sum_{0 \le d \le L_y, \ d \ even} Tr[(T_{Z,\Lambda,L_y,d})^m] - \sum_{1 \le d \le L_y, \ d \ odd} Tr[(T_{Z,\Lambda,L_y,d})^m] = 0 \quad \text{for} \quad q = 0 \ . \tag{3.9}$$

This is similar to a sum rule that we obtained in Ref. [21]. Since it applies for arbitrary m, it implies that there must be a pairwise cancellation between various eigenvalues in different degree-d subspaces, which, in turn, implies that at $q = 0$ there are equalities between these eigenvalues. For example, for $L_y = 1$ and $q = 0$, $\lambda_{Z,sq,1,0} = q+v$ becomes equal to $\lambda_{Z,sq,1,1} = v$. For the $L_y = 2$ strips of all of the $\Lambda = sq, tri, hc$ lattices, $q = 0$, two of the eigenvalues of $T_{Z,\Lambda,2,1}$ become equal to the eigenvalues of $T_{Z,\Lambda,2,0}$, while the third eigenvalue of $T_{Z,\Lambda,2,1}$ becomes equal to $\lambda_{Z,\Lambda,2,2} = v^{2p}$, and so forth for higher L_y. Note that setting $q = 0$ does not, in general, lead to any vanishing eigenvalues for the $T_{Z,\Lambda,L_y,d}$ and hence our formulas below for $T_{Z,\Lambda,L_y,d}$ do not contain overall factors of q.

In terms of Tutte polynomial parameters, the value $q = 0$ implies $x = 1$ (unless $v = 0$). In contrast to the vanishing of $Z(G, q, v)$ at $q = 0$, the Tutte polynomial $T(G, 1, y)$ is nonzero for general y. This different behavior can be traced to the feature that in eq. (1.12), $Z(G, q, v)$ is proportional to $T(G, x, y)$ multiplied by the factor $(x - 1)^{k(G)} = (x - 1)$, so at $x = 1$, $Z(G, 0, v) = 0$ even if $T(G, 1, y) \ne 0$.

C. $q = 1$

Evaluating eq. (1.1) for $q = 1$, one sees that the Kronecker delta functions $\delta_{\sigma_i \sigma_j} = 1$ for all pairs of adjacent vertices $\langle i, j \rangle$; consequently,

$$Z(G, 1, v) = e^{e(G)K} = (1 + v)^{e(G)} \ . \tag{3.10}$$

Now [12]

$$\text{If} \quad q = 1 \quad \text{then} \quad c^{(d)} = \begin{cases} 1 & \text{if } d = 0 \mod 3 \\ 0 & \text{if } d = 1 \mod 3 \\ -1 & \text{if } d = 2 \mod 3 \end{cases} \tag{3.11}$$

Hence, in terms of transfer matrices, we derive the sum rule for the present cyclic lattice strips $G = \Lambda, L_y \times L_x, cyc.$

$$\sum_{0 \le d \le L_y, d=0 \ mod \ 3} Tr[(T_{Z,\Lambda,L_y,d})^m] - \sum_{2 \le d \le L_y, d=2 \ mod \ 3} Tr[(T_{Z,\Lambda,L_y,d})^m] = (1+v)^{e(G)} \quad \text{for} \quad q = 1 \ , \tag{3.12}$$

Here the number of edges $e(G)$ for each type of cyclic strip is

$$e(G) = \begin{cases} (2L_y - 1)m & \text{if } \Lambda = sq \\ (3L_y - 2)m & \text{if } \Lambda = tri \\ (3L_y - 1)m & \text{if } \Lambda = hc \\ 2L_y m & \text{if } \Lambda = G_D \end{cases} \tag{3.13}$$

where m is given in terms of L_x by eq. (1.4). As before, the sum rule (3.12) implies relations between the eigenvalues of the various transfer matrices $T_{Z,\Lambda,L_y,d}$.

D. $v = -1$, i.e., $y = 0$

As discussed above, the special value $v = -1$, i.e., $y = 0$, corresponds to the zero-temperature Potts antiferromagnet, and in this case the Potts model partition function reduces to the chromatic polynomial as indicated in eq. (1.7). In Refs. [12,17,15] we have determined how the dimensions $n_Z(L_y, d)$ for the square, triangular, and honeycomb lattice strips and $n_Z(G_D, L_y, d)$ for the self-dual strip of the square-lattice reduce to dimensions $n_P(L_y, d)$ and $n_P(G_D, L_y, d)$, respectively. In all cases except for the lowest width $L_y = 1$ for the square lattice strip, in each degree-d subspace for $0 \le d \le L_y - 1$ for $\Lambda = sq, tri, hc$ and $1 \le d \le L_y$ for $\Lambda = G_D$, some eigenvalues vanish so that $n_P(\Lambda, L_y, d) < n_Z(\Lambda, L_y, d)$ and hence $det(T_{Z,\Lambda,L_y,d}) = 0$. This property is reflected in the powers of $(v + 1)$ and y that appear, respectively, in our formulas below for $det(T_{Z,\Lambda,L_y,d})$ and $det(T_{T,\Lambda,L_y,d})$.

E. $q=-v$, i.e., $x=0$

For the graph $G=G(V,E)$, setting $x=0$ and $y=1-q$ in the Tutte polynomial $T(G,x,y)$ yields the flow polynomial $F(G,q)$, which counts the number of nowhere-0 q-flows (without sinks or sources) that there are on G [10]:

$$F(G,q)=(-1)^{e(G)-n(G)+1}T(G,0,1-q) \ . \tag{3.14}$$

In Ref. [38] we showed that the flow polynomial for cyclic lattice strips has the form

$$F(\Lambda,L_y\times L_x,cyc.,q)=\sum_{d=0}^{L_y}c^{(d)}\sum_{j=1}^{n_F(\Lambda,L_y,d)}(\lambda_{F,\Lambda,L_y,d,j})^m \tag{3.15}$$

or equivalently,

$$F(\Lambda,L_y\times L_x,cyc.,q)=\sum_{d=0}^{L_y}c^{(d)}\sum_{j=1}^{n_F(\Lambda,L_y,d)}Tr[(T_{F,\Lambda,L_y,d,j})^m] \ . \tag{3.16}$$

We determined the dimensions $n_F(\Lambda,L_y,d)$ and found that for all $L_y\geq 1$, $n_F(\Lambda,L_y,d) < n_Z(\Lambda,L_y,d)$. Thus, when one sets $x=0$ in the Tutte polynomial, or equivalently, $q=-v$ in the Potts model partition function, some of the eigenvalues in each degree-d subspace vanish, and hence $det(T_{Z,\Lambda,L_y,d})$ and $det(T_{T,\Lambda,L_y,d})$ vanish. This is reflected in the powers of $(q+v)$ and x that appear, respectively, in our formulas below for $det(T_{Z,\Lambda,L_y,d})$ and $det(T_{T,\Lambda,L_y,d})$. In passing, we note that for planar graphs G_{pl}, because of the duality relation (1.13), the chromatic polynomial on G is closely related to the flow polynomial on G^*: $P(G,q)=qF(G^*,q)$.

These results concerning evaluations of the Potts model partition function on these lattice strips for special values of q,v, equivalent to evaluations of the Tutte polynomial for special values of x and y, help to explain the types of factors that appear in the simple formulas that we obtain below for the determinants $det(T_{Z,\Lambda,L_y,d})$ and $det(T_{T,\Lambda,L_y,d})$.

IV. GENERAL RESULTS FOR CYCLIC STRIPS OF THE SQUARE LATTICE

In this section and the subsequent ones we present general results that we have obtained, valid for arbitrarily large strip width L_y (as well as arbitrarily great length) for transfer matrices and their properties. We begin with the strip of the square lattice.

A. Determinants

We find

$$det(T_{T,sq,L_y,d}) = (x^{L_y} y^{L_y-1})^{n_Z(L_y-1,d)} \tag{4.1}$$

where $n_Z(L_y,d)$ was given in eq. (1.20). This applies for all d, i.e., $0 \le d \le L_y$ since [12] $n_Z(L_y-1,d) = 0$ for $d > L_y - 1$. This is equivalent to the somewhat more complicated expression for $det(T_{Z,sq,L_y,d})$:

$$det(T_{Z,sq,L_y,d}) = (v^{L_y})^{n_Z(L_y,d)} \Big[\Big(1+\frac{q}{v}\Big)^{L_y} (1+v)^{L_y-1} \Big]^{n_Z(L_y-1,d)} \quad . \tag{4.2}$$

These determinant formulas can be explained as follows. By the arrangement of the partitions as shown in Figs. 2 and 3, the matrix $J_{L_y,d,i,i+1}$ has the lower triangular form and the matrix $D_{L_y,d,i}$ has the upper triangular form. From the definition of the transfer matrix in eq. (2.8), the determinant of $T_{Z,sq,L_y,d}$ is the product of the diagonal elements of $H_{Z,sq,L_y,d}$ and $V_{Z,sq,L_y,d}$. The diagonal elements of $H_{Z,sq,L_y,d}$ has the form $(1+v)^r$, where r is the number of edges in the corresponding partition, and the diagonal elements of $V_{Z,sq,L_y,d}$ has the form $v^{L_y}(1+q/v)^s$, where s is the number of vertices which does not connect to any other vertex in the corresponding partition. Therefore, the power of $(1+v)$ in eq. (4.2) is the sum of the number of edges of all the (L_y,d)-partitions. Let us compare the (L_y,d)-partitions and (L_y-1,d) partitions. Since each edge of a (L_y,d)-partition corresponds to converting a vertex of a (L_y-1,d)-partition into a edge, the power of $(1+v)$ is $(L_y-1)n_Z(L_y-1,d)$. It is clear that the power of v is $L_y n_Z(L_y,d)$. The power of $(1+q/v)$ in eq. (4.2) is the sum of the number of unconnected vertices of all the (L_y,d)-partitions. Compare the (L_y,d)-partitions and (L_y-1,d) partitions again. Since each unconnected vertex of a (L_y,d)-partition corresponds to adding an unconnected vertex to a (L_y-1,d)-partition with L_y possible ways (between every two adjacent vertices, above the highest vertex, and below the lowest vertex), the power of $(1+q/v)$ is $L_y n_Z(L_y-1,d)$.

Next, taking into account that the generalized multiplicity of each $\lambda_{Z,\Lambda,L_y,d}$ is $c^{(d)}$, we have, for the total determinant

$$\begin{aligned} det(T_{Z,sq,L_y}) &= \prod_{d=0}^{L_y} [det(T_{Z,sq,L_y,d})]^{c^{(d)}} \\ &= \prod_{d=0}^{L_y} [(y-1)^{L_y}]^{n_Z(L_y,d)c^{(d)}} [x^{L_y} y^{L_y-1}]^{n_Z(L_y-1,d)c^{(d)}} \\ &= [(y-1)^{L_y}]^{\sum_{d=0}^{L_y} n_Z(L_y,d)c^{(d)}} [x^{L_y} y^{L_y-1}]^{\sum_{d=0}^{L_y} n_Z(L_y-1,d)c^{(d)}} \quad . \end{aligned} \tag{4.3}$$

Using eq. (2.19) together with eq. (4.2) from Ref. [12], viz., $n_Z(L_y,d)=0$ for $d>L_y$, so that $\sum_{d=0}^{L_y} n_Z(L_y-1,d)c^{(d)} = \sum_{d=0}^{L_y-1} n_Z(L_y-1,d)c^{(d)}$, we have, finally,

$$det(T_{Z,sq,L_y}) = [(y-1)^{L_y}]^{q^{L_y}} [x^{L_y} y^{L_y-1}]^{q^{L_y-1}} \ . \tag{4.4}$$

This agrees with the conjecture given as eq. (2.42) of our earlier Ref. [13] for the determinant of the transfer matrix of the cyclic strip of the square lattice with arbitrary width L_y.

B. Traces

For the trace of the total transfer matrix, taking account of the fact that each of the $\lambda_{X,\Lambda,d,j}$, $X=Z,T$, has multiplicity $c^{(d)}$, we have

$$\begin{aligned} Tr(T_{T,sq,L_y}) &= \frac{1}{x-1}\sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} \lambda_{T,sq,L_y,d,j} = \frac{1}{x-1}\sum_{d=0}^{L_y} c^{(d)} Tr(T_{T,sq,L_y,d}) \\ &= x^{L_y-1}y^{L_y} \ . \end{aligned} \tag{4.5}$$

Note that $\sum_{d=0}^{L_y} c^{(d)} Tr(T_{T,sq,L_y,d})$ contains a factor of $(x-1)$ which cancels the prefactor $1/(x-1)$. Equivalently, we find

$$\begin{aligned} Tr(T_{Z,sq,L_y}) &= \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} \lambda_{Z,sq,L_y,d,j} = \sum_{d=0}^{L_y} c^{(d)} Tr(T_{Z,sq,L_y,d}) \\ &= q(v+q)^{L_y-1}(1+v)^{L_y} \ . \end{aligned} \tag{4.6}$$

We derive this as follows. The trace here is the $m=1$ case in eq. (1.17) or (1.6), which corresponds to a L_y-vertex tree with a loop attached to each vertex, as shown in Fig. 4. The corresponding Tutte polynomial is $x^{L_y-1}y^{L_y}$. In principle, we can also consider the $m=1$ case in eq. (2.13) or (2.14) for the Möbius strips, but the result is not as simple as that listed here.

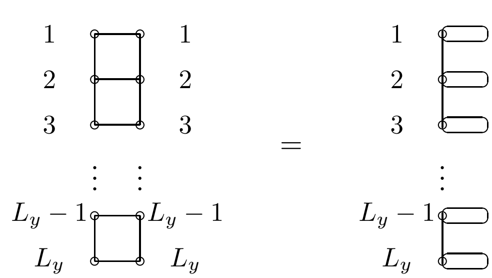

FIG. 4. $m=1$ graph for the cyclic square lattice.

C. Eigenvalue for $d = L_y$ for $\Lambda = sq, tri, hc$

It was shown earlier [11,39] that the λ's are the same for a given lattice strip with cyclic, as compared with Möbius, boundary conditions (and it was shown that the λ's for a strip with Klein bottle boundary conditions are a subset of the λ's for the same strip with torus boundary conditions). From eq. (1.20) one knows that there is only one λ for degree $d = L_y$, which we denote $\lambda_{Z,\Lambda,L_y,L_y}$. That is, for this value of d, the transfer matrix reduces to 1×1, i.e. a scalar. We found that for a cyclic or Möbius strip of the square, triangular, or honeycomb lattice with width L_y,

$$\lambda_{T,\Lambda,L_y,L_y} = 1 \ . \tag{4.7}$$

(Indeed, we have found that this holds more generally for other strips of regular lattices (e.g., [19]), but we shall not pursue this here.) Note the universality; i.e., this is the same for all of these lattices. In terms of Potts model variables the results do not maintain the full universality:

$$\lambda_{Z,\Lambda,L_y,L_y} = v^{L_y} \quad \text{for} \quad \Lambda = sq, tri \tag{4.8}$$

$$\lambda_{Z,\Lambda,L_y,L_y} = v^{2L_y} \quad \text{for} \quad \Lambda = hc \ . \tag{4.9}$$

D. Transfer Matrix for $d = L_y - 1$, $\Lambda = sq$

From eq. (1.20) it follows that for $\Lambda = sq, tri$ or hc, the number of $\lambda_{X,L_y,d,j}$, $X = Z$ or $X = T$, for $d = L_y - 1$ is $n_Z(L_y, L_y - 1) = 2L_y - 1$, i.e. the transfer matrix in this subspace, $T_{X,\Lambda,d}$, $X = Z$ or $X = T$, is a (square) $(2L_y - 1)$-dimensional matrix. We recall that for $L_y = 1$, $T_{T,sq,1,0} = x$. For $L_y \geq 2$ we find the following general formula, which we write for the Tutte polynomial, since it has a somewhat simpler form.

$$(T_{T,sq,L_y,L_y-1})_{j,j} = 1 + x \quad \text{for} \quad j = 1 \ \text{ and } \ j = L_y \tag{4.10}$$

$$(T_{T,sq,L_y,L_y-1})_{j,j} = 2 + x \quad \text{for} \quad L_y \geq 3 \ \text{ and } \ 2 \leq j \leq L_y - 1 \tag{4.11}$$

$$(T_{T,sq,L_y,L_y-1})_{j,j+1} = (T_{T,sq,L_y,L_y-1})_{j+1,j} = 1 \quad \text{for} \quad 1 \leq j \leq L_y - 1 \tag{4.12}$$

$$(T_{T,sq,L_y,L_y-1})_{j,j} = y \quad \text{for} \quad L + 1 \leq j \leq 2L_y - 1 \tag{4.13}$$

$$(T_{T,sq,L_y,L_y-1})_{j,L_y+j} = (T_{T,sq,L_y,L_y-1})_{j+1,L_y+j} = \frac{y}{y-1} \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{4.14}$$

$$(T_{T,sq,L_y,L_y-1})_{L_y+j,j} = (T_{T,sq,L_y,L_y-1})_{L_y+j,j+1} = y - 1 \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{4.15}$$

with all other elements equal to zero. The elements of T_{Z,sq,L_y-1} are given by the relation (2.27). Thus, T_{X,sq,L_y,L_y-1} for $X = T, Z$ consists of four submatrices:

1. an upper left square submatrix with indices i, j in the ranges $1 \le i, j \le L_y$ and nonzero elements given by eqs. (4.10)-(4.12)

2. a lower right square submatrix with indices in the ranges $L_y + 1 \le i, j \le 2L_y - 1$ and nonzero elements given by eqs. (4.13)

3. an upper right rectangular submatrix with nonzero elements given by eq. (4.14)

4. a lower left rectangular submatrix with nonzero elements given by eq. (4.15).

For general x and y, T_{T,sq,L_y,L_y-1} has rank equal to its dimension, $2L_y - 1$. In the special case $y = 0$ which yields the chromatic polynomial, the rank is reduced to L_y for general x (and may be reduced further for particular x). In the special case $x = 0$ which yields the flow polynomial, the rank is reduced to $L_y - 1$ (and may be reduced further for particular y).

We illustrate these general formulas for the cases $L_y = 2, 3, 4$. For this purpose we introduce the abbreviations

$$x_1 = 1 + x, \quad x_2 = 2 + x, \quad r_d = \frac{y}{y-1} \tag{4.16}$$

and use $v = y - 1$ as before.

$$T_{T,sq,2,1} = \begin{pmatrix} x_1 & 1 & r_d \\ 1 & x_1 & r_d \\ v & v & y \end{pmatrix} \tag{4.17}$$

$$T_{T,sq,3,2} = \begin{pmatrix} x_1 & 1 & 0 & r_d & 0 \\ 1 & x_2 & 1 & r_d & r_d \\ 0 & 1 & x_1 & 0 & r_d \\ v & v & 0 & y & 0 \\ 0 & v & v & 0 & y \end{pmatrix} \tag{4.18}$$

$$
T_{T,sq,4,3} = \begin{pmatrix} x_1 & 1 & 0 & 0 & r_d & 0 & 0 \\ 1 & x_2 & 1 & 0 & r_d & r_d & 0 \\ 0 & 1 & x_2 & 1 & 0 & r_d & r_d \\ 0 & 0 & 1 & x_1 & 0 & 0 & r_d \\ v & v & 0 & 0 & y & 0 & 0 \\ 0 & v & v & 0 & 0 & y & 0 \\ 0 & 0 & v & v & 0 & 0 & y \end{pmatrix} \tag{4.19}
$$

Thus, in general, the upper left-hand submatrix has a main diagonal with end entries equal to x_1 and interior entries equal to x_2. Adjacent to this main diagonal are two diagonals whose entries are 1, and the rest of the submatrix is comprised of triangular regions filled with 0's. The upper right-hand submatrix has a band of two diagonals whose entries are r_d together with triangular regions filled with 0's. The lower left-hand submatrix has a band of two diagonals whose entries are v together with triangular regions filled with 0's. And finally, in the right-hand lower submatrix the entries on the main diagonal are equal to y and the rest of this submatrix is made up of triangular regions of 0's. For the lowest values $L_y = 1, 2$, some of these parts, such as the triangular regions of zeros, are not present.

Using eq. (2.27), it is straightforward to obtain the T_{Z,sq,L_y,L_y-1} from these T_{T,sq,L_y,L_y-1} matrices. For example,

$$
T_{Z,sq,2,1} = \begin{pmatrix} v(q+2v) & v^2 & v(1+v) \\ v^2 & v(q+2v) & v(1+v) \\ v^3 & v^3 & v^2(1+v) \end{pmatrix} \tag{4.20}
$$

Having determined the general form of $T_{T,sq,L_y,d}$ for $d = L_y - 1$, we calculate its eigenvalues $\lambda_{T,sq,L_y,L_y-1,j}$. We find that these consist of one that is the same independent of L_y, namely

$$
\lambda_{T,sq,L_y,L_y-1,1} = x \tag{4.21}
$$

together with $2(L_y - 1)$ quadratic roots. It is convenient to label these as $(j, \pm)$ with $2 \le j \le L_y$. We find

$$
\lambda_{T,sq,L_y,L_y-1,j,\pm} = \frac{1}{2}\Big[x + y + b_{sq,L_y,j} \pm \sqrt{(x+y+b_{sq,L_y,j})^2 - 4xy}\ \Big] \quad \text{for} \quad 2 \le j \le L_y \tag{4.22}
$$

where

$$
b_{sq,L_y,j} \equiv a_{sq,L_y,j} - 1 = 4\cos^2\Big(\frac{(L_y+1-j)\pi}{2L_y}\Big) \ . \tag{4.23}
$$

Note that the product $(\lambda_{T,sq,L_y,L_y-1,j,+})(\lambda_{T,sq,L_y,L_y-1,j,-})$ is independent of j:

$$
(\lambda_{T,sq,L_y,L_y-1,j,+})(\lambda_{T,sq,L_y,L_y-1,j,-}) = xy \ , \quad \text{for} \quad 2 \le j \le L_y \ . \tag{4.24}
$$

This generalizes to the Tutte polynomial our determination of the L_y-1 terms $\lambda_{P,sq,L_y,L_y-1,j}$ for the chromatic polynomial in eqs. (7.1.2) and (7.1.3) of Ref. [21], and $a_{sq,L_y,j}$ was given in eq. (7.1.3) of that paper. Note that in the special case $y=0$ ($v=-1$) in which the Tutte polynomial or Potts model partition function reduces to the chromatic polynomial, one of each of the L_y-1 pairs of quadratic roots vanishes and the other becomes $x+b_{sq,L_y,j}$ for the Tutte polynomial, or equivalently, $(-1)^{L_y}(a_{sq,L_y,j}-q)$ for the Potts model, in agreement with eq. (7.1.2) of Ref. [21]. The term $(-1)^{L_y}(a_{sq,L_y,1}-q)=(-1)^{L_y}(1-q)$ in eq. (7.1.2) of Ref. [21] corresponds to the $v=-1$ special case of eq. (4.21).

As corollaries of our general result for T_{T,sq,L_y,L_y-1} we calculate the trace and determinant. For this purpose, we note that

$$\sum_{j=2}^{L_y} b_{sq,L_y,j} = 4\sum_{j=2}^{L_y}\cos^2\Big(\frac{(L_y+1-j)\pi}{2L_y}\Big) = 2(L_y-1) \tag{4.25}$$

$$det(T_{T,sq,L_y,L_y-1}) = x^{L_y}y^{L_y-1} \tag{4.26}$$

which is a special case of eq. (4.1), and

$$Tr(T_{T,sq,L_y,L_y-1}) = x+(L_y-1)(2+x+y) = (L_y-1)(2+y)+L_y x \ . \tag{4.27}$$

In terms of Potts model quantities these results are

$$\lambda_{Z,sq,L_y,L_y-1,1} = v^{L_y-1}(v+q) \tag{4.28}$$

$$\lambda_{Z,sq,L_y,L_y-1,j\pm} = \frac{v^{L_y-1}}{2}\Big[q+v(v+b_{sq,L_y,j}+2) \pm \sqrt{[q+v(v+b_{sq,L_y,j}+2)]^2-4v(v+q)(v+1)}\ \Big]$$
$$\text{for} \quad 2 \le j \le L_y \tag{4.29}$$

$$det(T_{Z,sq,L_y,L_y-1}) = v^{2L_y(L_y-1)}(v+q)^{L_y}(v+1)^{L_y-1} \tag{4.30}$$

and

$$Tr(T_{Z,sq,L_y,L_y-1}) = v^{L_y-1}\Big[(4L_y-3)v+(L_y-1)v^2+L_y q\Big] \ . \tag{4.31}$$

V. GENERAL RESULTS FOR CYCLIC STRIPS OF THE TRIANGULAR LATTICE

A. Determinants

We find

$$det(T_{T,tri,L_y,d}) = (x^{L_y} y^{2(L_y-1)})^{n_Z(L_y-1,d)} \ . \tag{5.1}$$

Equivalently,

$$det(T_{Z,tri,L_y,d}) = (v^{L_y})^{n_Z(L_y,d)} \Big[\Big(1+\frac{q}{v}\Big)^{L_y} (1+v)^{2(L_y-1)} \Big]^{n_Z(L_y-1,d)} \tag{5.2}$$

By the same argument as before, the powers of v and $(1+q/v)$ are equal to the corresponding powers for the square lattice case. Comparing $V_{Z,sq,L_y,d}$ and $V_{Z,tri,L_y,d}$ in eq. (2.7), one sees that a set of $(I+vJ_{L_y,d,i,i+1})$ has been included for the triangular lattice, so that the power of $(1+v)$ becomes twice of the corresponding power for the square lattice.

Taking into account that the multiplicity of each $\lambda_{X,\Lambda,L_y,d,j}$, $X=Z,T$ is $c^{(d)}$, it follows that the total determinant is

$$det(T_{T,tri,L_y}) \equiv \prod_{d=0}^{L_y} [det(T_{T,tri,L_y,d})]^{c^{(d)}} = (x^{L_y} y^{2(L_y-1)})^{q^{L_y-1}} \ . \tag{5.3}$$

Equivalently,

$$det(T_{Z,tri,L_y}) \equiv \prod_{d=0}^{L_y} [det(T_{Z,tri,L_y,d})]^{c^{(d)}} = (v^{L_y})^{q^{L_y}} \Big[\Big(1+\frac{q}{v}\Big)^{L_y} (1+v)^{2(L_y-1)} \Big]^{q^{L_y-1}} \ . \tag{5.4}$$

B. Traces

For the total trace, taking account of the fact that each of the $\lambda_{X,\Lambda,d,j}$ has multiplicity $c^{(d)}$, we have

$$\begin{aligned} Tr(T_{T,tri,L_y}) &= \frac{1}{x-1} \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} \lambda_{T,tri,L_y,d,j} = \frac{1}{x-1} \sum_{d=0}^{L_y} c^{(d)} Tr(T_{T,tri,L_y,d}) \\ &= (x+y)^{L_y-1} y^{L_y} \ . \end{aligned} \tag{5.5}$$

Equivalently,

$$Tr(T_{Z,tri,L_y}) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} \lambda_{Z,sq,L_y,d,j} = \sum_{d=0}^{L_y} c^{(d)} Tr(T_{Z,tri,L_y,d})$$
$$= q(q+2v+v^2)^{L_y-1}(v+1)^{L_y} \ . \tag{5.6}$$

Again the trace here is related to the $m=1$ case in eq. (1.17) or (1.6), which corresponds to a L_y-vertex tree with each edge doubled and with a loop attached to each vertex as shown in Fig. 5. Therefore, the Tutte polynomial of this graph is $(x+y)^{L_y-1}y^{L_y}$.

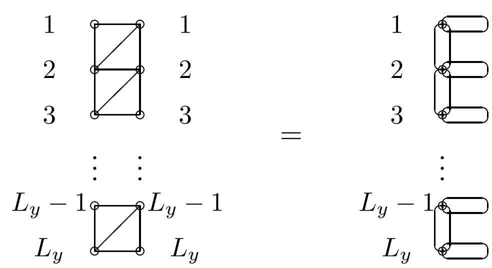

FIG. 5. $m=1$ graph for the cyclic triangular lattice.

C. Transfer Matrix for $d = L_y - 1$, $\Lambda = tri$

As before, from eq. (1.20), we know that the dimension of this transfer matrix is $2L_y-1$. The first nontrivial case is $L_y = 2$. For $L_y \ge 2$ we find the following general formula, which we write for the Tutte polynomial, since it has a somewhat simpler form.

$$(T_{T,tri,L_y,L_y-1})_{1,1} = 2+x+y \tag{5.7}$$

$$(T_{T,tri,L_y,L_y-1})_{j,j} = 3+x+y \quad \text{for} \quad 2 \le j \le L_y-1 \tag{5.8}$$

$$(T_{T,tri,L_y,L_y-1})_{L_y,L_y} = 1+x \tag{5.9}$$

$$(T_{T,tri,L_y,L_y-1})_{1+j,j} = 1 \quad \text{for} \quad 1 \le j \le L_y-1 \tag{5.10}$$

$$(T_{T,tri,L_y,L_y-1})_{j,k} = 3+x+2y \quad \text{for} \quad 1 \le j \le L_y-2 \quad \text{and} \quad j+1 \le k \le L_y-1 \tag{5.11}$$

$$(T_{T,tri,L_y,L_y-1})_{j,L_y} = 1+x+y \quad \text{for} \quad 1 \le j \le L_y-1 \tag{5.12}$$

$$(T_{T,tri,L_y,L_y-1})_{1+j,L_y+j} = \frac{y}{y-1} \quad \text{for} \quad 1 \le j \le L_y-1 \tag{5.13}$$

$$(T_{T,tri,L_y,L_y-1})_{j,L_y+k} = \frac{y(1+y)}{y-1} \quad \text{for} \quad 1 \le j \le L_y - 1 \quad \text{and} \quad j \le k \le L_y - 1 \tag{5.14}$$

$$(T_{T,tri,L_y,L_y-1})_{j,k} = (y-1)(2+y) \quad \text{for} \quad 1 \le j \le L_y - 2 \quad \text{and} \quad j+1 \le k \le L_y - 1 \tag{5.15}$$

$$(T_{T,tri,L_y,L_y-1})_{L_y+j,1+k} = (y-1)(3+x+2y) \quad \text{for} \quad 1 \le j \le L_y - 2 \quad \text{and} \quad j \le k \le L_y - 1 \tag{5.16}$$

$$(T_{T,tri,L_y,L_y-1})_{L_y,j} = (y-1)(1+x+y) \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{5.17}$$

$$(T_{T,tri,L_y,L_y-1})_{j,L_y+k} = y(1+y) \quad \text{for} \quad 1 \le j \le k \le L_y - 1 \tag{5.18}$$

with all other elements equal to zero. The elements of T_{Z,tri,L_y,L_y-1} are given by the relation (2.27). Thus, T_{X,tri,L_y,L_y-1} can again be usefully viewed as consisting of various submatrices.

For general x and y, T_{T,tri,L_y,L_y-1} has rank equal to its dimension, $2L_y - 1$. In the special case $y = 0$ (chromatic polynomial), the rank is reduced to L_y for general x (and may be reduced further for particular x). In the special case $x = 0$ (flow polynomial), the rank is reduced to $2(L_y - 1)$ (and may be reduced further for particular y).

We illustrate these general formulas with some explicit examples for $L_y = 2, 3, 4$. For compactness of notation, we use the abbreviations

$$t_{11} = 2 + x + y, \quad l_{ud} = 3 + x + y, \qquad l_u = 3 + x + 2y, \tag{5.19}$$

$$l_l = (y-1)l_u = (y-1)(3+x+2y), \quad l_d = (y-1)(y+2), \tag{5.20}$$

$$c_u = 1 + x + y, \quad c_l = (y-1)c_u = (y-1)(1+x+y), \tag{5.21}$$

$$r_u = \frac{y(y+1)}{y-1}, \quad r_l = y(y+1) \tag{5.22}$$

(with $x_1 = 1 + x$ and $r_d = y/(y-1)$ as above). Then

$$T_{T,tri,2,1} = \begin{pmatrix} t_{11} & c_u & r_u \\ 1 & x_1 & r_d \\ l_d & c_l & r_l \end{pmatrix} \tag{5.23}$$

$$T_{T,tri,3,2} = \begin{pmatrix} t_{11} & l_u & c_u & r_u & r_u \\ 1 & l_{ud} & c_u & r_d & r_u \\ 0 & 1 & x_1 & 0 & r_d \\ l_d & l_l & c_l & r_l & r_l \\ 0 & l_d & c_l & 0 & r_l \end{pmatrix} \tag{5.24}$$

$$T_{T,tri,4,3} = \begin{pmatrix} t_{11} & l_u & l_u & c_u & r_u & r_u & r_u \\ 1 & l_{ud} & l_u & c_u & r_d & r_u & r_u \\ 0 & 1 & l_{ud} & c_u & 0 & r_d & r_u \\ 0 & 0 & 1 & x_1 & 0 & 0 & r_d \\ l_d & l_l & l_l & c_l & r_l & r_l & r_l \\ 0 & l_d & l_l & c_l & 0 & r_l & r_l \\ 0 & 0 & l_d & c_l & 0 & 0 & r_l \end{pmatrix} \tag{5.25}$$

It is straightforward to obtain the T_{Z,tri,L_y,L_y-1} from these T_{T,tri,L_y,L_y-1} matrices; for example,

$$T_{Z,tri,2,1} = \begin{pmatrix} v(q+4v+v^2) & v(q+3v+v^2) & v(2+v)(1+v) \\ v^2 & v(q+2v) & v(1+v) \\ v^3(3+v) & v^2(q+3v+v^2) & v^2(2+v)(1+v) \end{pmatrix} \tag{5.26}$$

VI. GENERAL RESULTS FOR CYCLIC STRIPS OF THE HONEYCOMB LATTICE

A. Determinants

We find

$$det(T_{T,hc,L_y,d}) = (x^{2L_y} y^{L_y-1})^{n_Z(L_y-1,d)} \ . \tag{6.1}$$

Equivalently,

$$det(T_{Z,hc,L_y,d}) = (v^{2L_y})^{n_Z(L_y,d)} \Big[\Big(1+\frac{q}{v}\Big)^{2L_y} (1+v)^{L_y-1} \Big]^{n_Z(L_y-1,d)} \ . \tag{6.2}$$

This can be understood as follows: by an argument similar to that given before, the power of $(1+v)$ is the same as for the square lattice case. Comparing $T_{Z,sq,L_y,d}$ and $T_{Z,hc,L_y,d}$ in eq. (2.8), one sees that $V_{Z,hc,L_y,d} = V_{Z,sq,L_y,d}$ has been multiplied twice for the honeycomb lattice, so that the powers of v and $(1+q/v)$ become twice of the corresponding powers for the square lattice.

Taking into account that the multiplicity of each $\lambda_{X,\Lambda,L_y,d,j}$, $X = Z,T$ is $c^{(d)}$, it follows that the total determinant for the hc lattice is

$$det(T_{T,hc,L_y}) \equiv \prod_{d=0}^{L_y} [det(T_{T,hc,L_y,d})]^{c^{(d)}} = (x^{2L_y} y^{L_y-1})^{q^{L_y-1}} \ . \tag{6.3}$$

Equivalently,

$$det(T_{Z,hc,L_y}) \equiv \prod_{d=0}^{L_y} [det(T_{Z,hc,L_y,d})]^{c^{(d)}} = (v^{2L_y})^{q^{L_y}} \left[\left(1+\frac{q}{v}\right)^{2L_y} (1+v)^{L_y-1} \right]^{q^{L_y-1}} \ . \tag{6.4}$$

Summarizing the connections between the determinants of the transfer matrices for the three lattice strips, $det(T_{T,tri,L_y,d})$ is related to $det(T_{T,sq,L_y,d})$ by the replacement $y \to y^2$ (holding x fixed), while $det(T_{T,hc,L_y,d})$ is related to $det(T_{T,sq,L_y,d})$ by the replacement $x \to x^2$ (holding y fixed). This, together with the fact that $n_Z(L_y, d)$ is the same for all of these three lattices means that the total determinants $det(T_{T,tri,L_y})$ and $det(T_{T,hc,L_y})$ are related to $det(T_{T,sq,L_y})$ by the same respective replacements. Correspondingly, $det(T_{Z,tri,L_y,d})$ is related to $det(T_{Z,sq,L_y,d})$ by the replacement of $(1+v)$ by $(1+v)^2$ and $det(T_{Z,hc,L_y,d})$ is related to $det(T_{Z,sq,L_y,d})$ by the replacements of the respective factors v by v^2 (cf. eq. (2.28)) and $(1+\frac{q}{v})$ by $(1+\frac{q}{v})^2$.

B. Traces

For the total trace, taking account of the fact that each of the $\lambda_{X,\Lambda,d,j}$ has multiplicity $c^{(d)}$, we have

$$\begin{aligned} Tr(T_{T,hc,L_y}) &= \frac{1}{x-1} \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} \lambda_{T,hc,L_y,d,j} = \frac{1}{x-1} \sum_{d=0}^{L_y} c^{(d)} Tr(T_{T,hc,L_y,d}) \\ &= x^{L_y-1}(x+y)^{L_y} \ . \end{aligned} \tag{6.5}$$

Equivalently,

$$\begin{aligned} Tr(T_{Z,hc,L_y}) &= \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} \lambda_{Z,hc,L_y,d,j} = \sum_{d=0}^{L_y} c^{(d)} Tr(T_{Z,hc,L_y,d}) \\ &= q(q+v)^{L_y-1}(q+2v+v^2)^{L_y} \end{aligned} \tag{6.6}$$

The trace here is related to the $m=1$ case in eq. (1.17) or (1.6), which corresponds to a $2L_y$-vertex tree with each odd edge doubled as shown in Fig. 6. Therefore, the Tutte polynomial of this graph is $x^{L_y-1}(x+y)^{L_y}$.

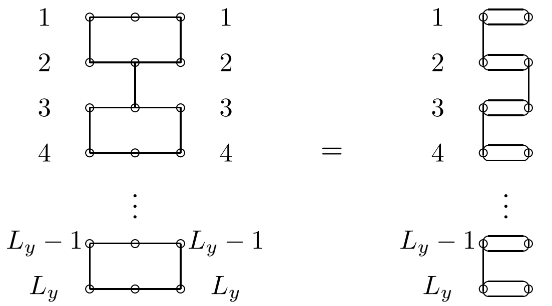

FIG. 6. $m=1$ graph for the cyclic honeycomb lattice with even L_y.

C. Transfer Matrix for $d = L_y - 1$, $\Lambda = hc$

As before, from eq. (1.20), we know that the dimension of this transfer matrix is $2L_y - 1$. The first nontrivial case is $L_y = 2$. Recall in eq. (2.8) the transfer matrix for the honeycomb lattice, T_{Z,hc,L_y,L_y-1}, is the product of $T_{Z,hc,L_y,L_y-1,1}$ and $T_{Z,hc,L_y,L_y-1,2}$. For $L_y \ge 2$ we find the following general formula, which we write for the Tutte polynomial.

$$(T_{T,hc,L_y,L_y-1,1})_{j,j} = 1 + x \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{6.7}$$

$$(T_{T,hc,L_y,L_y-1,1})_{L_y,L_y} = \begin{cases} 1 + x & \text{for } L_y \text{ even} \\ x & \text{for } L_y \text{ odd} \end{cases} \tag{6.8}$$

$$(T_{T,hc,L_y,L_y-1,1})_{2j-1,2j} = (T_{T,hc,L_y,L_y-1,1})_{2j,2j-1} = 1 \quad \text{for} \quad 1 \le j \le [L_y/2] \tag{6.9}$$

$$(T_{T,hc,L_y,L_y-1,1})_{L_y+2j-1,L_y+2j-1} = y \quad \text{for} \quad 1 \le j \le [L_y/2] \tag{6.10}$$

$$(T_{T,hc,L_y,L_y-1,1})_{L_y+2j,L_y+2j} = 1 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \tag{6.11}$$

$$(T_{T,hc,L_y,L_y-1,1})_{2j-1,L_y+2j-1} = (T_{T,hc,L_y,L_y-1,1})_{2j,L_y+2j-1} = \frac{y}{y-1} \quad \text{for} \quad 1 \le j \le [L_y/2] \tag{6.12}$$

$$(T_{T,hc,L_y,L_y-1,1})_{2j,L_y+2j} = (T_{T,hc,L_y,L_y-1,1})_{2j+1,L_y+2j} = \frac{1}{y-1} \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \tag{6.13}$$

$$(T_{T,hc,L_y,L_y-1,1})_{L_y+2j-1,2j-1} = (T_{T,hc,L_y,L_y-1,1})_{L_y+2j-1,2j} = y - 1 \quad \text{for} \quad 1 \le j \le [L_y/2] \tag{6.14}$$

$$(T_{T,hc,L_y,L_y-1,2})_{1,1} = x \tag{6.15}$$

$$(T_{T,hc,L_y,L_y-1,2})_{j,j} = 1+x \quad \text{for} \quad 2 \le j \le L_y - 1 \tag{6.16}$$

$$(T_{T,hc,L_y,L_y-1,2})_{L_y,L_y} = \begin{cases} 1+x & \text{for } L_y \text{ odd} \\ x & \text{for } L_y \text{ even} \end{cases} \tag{6.17}$$

$$(T_{T,hc,L_y,L_y-1,2})_{2j,2j+1} = (T_{T,hc,L_y,L_y-1,1})_{2j+1,2j} = 1 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \tag{6.18}$$

$$(T_{T,hc,L_y,L_y-1,2})_{L_y+2j-1,L_y+2j-1} = 1 \quad \text{for} \quad 1 \le j \le [L_y/2] \tag{6.19}$$

$$(T_{T,hc,L_y,L_y-1,2})_{L_y+2j,L_y+2j} = y \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \tag{6.20}$$

$$(T_{T,hc,L_y,L_y-1,2})_{2j-1,L_y+2j-1} = (T_{T,hc,L_y,L_y-1,2})_{2j,L_y+2j-1} = \frac{1}{y-1} \quad \text{for} \quad 1 \le j \le [L_y/2] \tag{6.21}$$

$$(T_{T,hc,L_y,L_y-1,2})_{2j,L_y+2j} = (T_{T,hc,L_y,L_y-1,2})_{2j+1,L_y+2j} = \frac{y}{y-1} \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \tag{6.22}$$

$$(T_{T,hc,L_y,L_y-1,2})_{L_y+2j,2j} = (T_{T,hc,L_y,L_y-1,1})_{L_y+2j,2j+1} = y-1 \quad \text{for} \quad 1 \le j \le [(L_y-1)/2] \tag{6.23}$$

with all other elements equal to zero.

For general x and y, T_{T,hc,L_y,L_y-1} has rank equal to its dimension, $2L_y - 1$. In the special case $y = 0$ (chromatic polynomial), the rank is reduced to $(3L_y - 1)/2$ for odd L_y and to $(3/2)L_y - 1$ for even L_y (and may be reduced further for particular x). In the special case $x = 0$ (flow polynomial), the rank is reduced to $L_y - 1$ (and may be reduced further for particular y). These results are in accord with our previous general findings that [12] $n_P(sq, L_y, d) = n_P(tri, L_y, d)$ and [15] $n_F(sq, L_y, d) = n_F(hc, L_y, d)$.

We illustrate these general formulas for the cases $L_y = 2, 3, 4$. For this purpose we use the abbreviations $x_1 = 1 + x$, $x_2 = 2 + x$ and $r_d = y/(y-1)$ as before.

$$T_{T,hc,2,1,1} = \begin{pmatrix} x_1 & 1 & r_d \\ 1 & x_1 & r_d \\ v & v & y \end{pmatrix} = T_{T,sq,2,1} \ , \qquad T_{T,hc,2,1,2} = \begin{pmatrix} x & 0 & 1/v \\ 0 & x & 1/v \\ 0 & 0 & 1 \end{pmatrix} \tag{6.24}$$

$$T_{T,hc,3,2,1} = \begin{pmatrix} x_1 & 1 & 0 & r_d & 0 \\ 1 & x_1 & 0 & r_d & 1/v \\ 0 & 0 & x & 0 & 1/v \\ v & v & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \ , \qquad T_{T,hc,3,2,2} = \begin{pmatrix} x & 0 & 0 & 1/v & 0 \\ 0 & x_1 & 1 & 1/v & r_d \\ 0 & 1 & x_1 & 0 & r_d \\ 0 & 0 & 0 & 1 & 0 \\ 0 & v & v & 0 & y \end{pmatrix} \tag{6.25}$$

$$T_{T,hc,4,3,1} = \begin{pmatrix} x_1 & 1 & 0 & 0 & r_d & 0 & 0 \\ 1 & x_1 & 0 & 0 & r_d & 1/v & 0 \\ 0 & 0 & x_1 & 1 & 0 & 1/v & r_d \\ 0 & 0 & 1 & x_1 & 0 & 0 & r_d \\ v & v & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & v & v & 0 & 0 & y \end{pmatrix}$$

$$T_{T,hc,4,3,2} = \begin{pmatrix} x & 0 & 0 & 0 & 1/v & 0 & 0 \\ 0 & x_1 & 1 & 0 & 1/v & r_d & 0 \\ 0 & 1 & x_1 & 0 & 0 & r_d & 1/v \\ 0 & 0 & 0 & x & 0 & 0 & 1/v \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & v & v & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.26}$$

The T_{X,hc,L_y,L_y-1}, $X = T, Z$ are obtained via eq. (2.8) from these auxiliary matrices. For example,

$$T_{T,hc,2,1} = \begin{pmatrix} 1+x+x^2 & 1+x & (1+x)y(y-1)^{-1} \\ 1+x & 1+x+x^2 & (1+x)y(y-1)^{-1} \\ y-1 & y-1 & y \end{pmatrix} \tag{6.27}$$

or equivalently,

$$T_{Z,hc,2,1} = \begin{pmatrix} v^2(3v^2+3vq+q^2) & v^3(2v+q) & v^2(2v+q)(1+v) \\ v^3(2v+q) & v^2(3v^2+3vq+q^2) & v^2(2v+q)(1+v) \\ v^5 & v^5 & v^4(1+v) \end{pmatrix} \tag{6.28}$$

We find that in general, for degree $d = L_y - 1$, one of the $2L_y - 1$ eigenvalues, say that for $j = 2L_y - 1$, has a particularly simple expression:

$$\lambda_{T,hc,L_y,d=L_y-1,j=2L_y-1} = x^2 \tag{6.29}$$

or equivalently,

$$\lambda_{Z,hc,L_y,d=L_y-1,j=2L_y-1} = v^{2(L_y-1)}(v+q)^2 \ . \tag{6.30}$$

The expressions for the other $2L_y$ eigenvalues are, in general, more complicated.

VII. GENERAL RESULTS FOR CYCLIC SELF-DUAL SQUARE-LATTICE STRIPS

In this section we consider the Potts model for families of self-dual strip graphs of the square lattice with fixed width L_y and arbitrarily great length L_x, having periodic longitudinal boundary conditions, such that all vertices on one side of the strip, which we take to be the upper side are joined by edges to a single external vertex. A strip graph of this type will be denoted generically as G_D and, in more detail, as $G_D(L_y \times L_x)$. The family of G_D graphs is planar and self-dual. In general, the graph $G_D(L_y \times L_x)$ has $n \equiv |V| = L_x L_y + 1$ vertices, equal to the number of faces, f. One motivation for considering the G_D strip graphs is that they exhibit, for any L_y, the self-duality property of the infinite square lattice so that, by (1.13),

$$T(G_D, x, y) = T(G_D, y, x) \ . \tag{7.1}$$

Equivalently, by (1.14), aside from a prefactor, the partition function is invariant under $v \to q/v$. This important duality property is used to derive the formula for the transition temperature of the q-state Potts ferromagnet, which occurs at the self-dual point $v = \sqrt{q}$.

In Ref. [17] we gave the general form for $Z(G_D, L_y \times L_x, q, v)$ which, in our current notation with m given in terms of L_x by eq. (1.4), is

$$Z(G_D, L_y \times L_x, q, v) = \sum_{d=1}^{L_y+1} \kappa^{(d)} Tr[(T_{Z,G_D,L_y,d})^m] \tag{7.2}$$

or equivalently

$$T(G_D, L_y \times L_x, x, y) = q^{-1} \sum_{d=1}^{L_y+1} \kappa^{(d)} Tr[(T_{T,G_D,L_y,d})^m] \tag{7.3}$$

where the factor of q^{-1} arising from a prefactor of $1/(x-1)$, and

$$\begin{aligned} \kappa^{(d)} &= \sqrt{q}\; U_{2d-1}\Big(\frac{\sqrt{q}}{2}\Big) \\ &= \sum_{j=0}^{d-1} (-1)^j \binom{2d-1-j}{j} q^{d-j} \ . \end{aligned} \tag{7.4}$$

The first few of these coefficients are

$$\kappa^{(1)} = q, \quad \kappa^{(2)} = q(q-2), \quad \kappa^{(3)} = q(q-1)(q-3), \quad \kappa^{(4)} = q(q-2)(q^2-4q+2) \ . \tag{7.5}$$

One has

$$\kappa^{(d)} = \prod_{k=1}^{d}(q - s_{d,k}) \tag{7.6}$$

where

$$s_{d,k} = 4\cos^2\left(\frac{\pi k}{2d}\right) \quad \text{for } k = 1, 2, ..d \tag{7.7}$$

and

$$\kappa^{(d)} = c^{(d)} + c^{(d-1)} \quad \text{for } d = 1, 2, ... \tag{7.8}$$

The dimension $dim(T_{Z,G_D,L_y,d}) = n_Z(G_D, L_y, d)$ was calculated in Ref. [17]; two relevant results are

$$n_Z(G_D, L_y, L_y) = 2L_y \ , \quad n_Z(G_D, L_y, L_y + 1) = 1 \ . \tag{7.9}$$

Combining eq. (7.1), (7.3), and the fact that $\kappa^{(d)}$ is a function of q which, in turn, is a symmetric function under $x \leftrightarrow y$ (cf. eq. (1.11)), it follows that the traces of the m'th powers of the transfer matrices themselves and their eigenvalues also have this symmetry:

$$Tr[(T_{G_D,L_y,d}(x,y))^m] = Tr[(T_{G_D,L_y,d}(y,x))^m] \tag{7.10}$$

$$\lambda_{T,G_D,L_y,d}(x,y) = \lambda_{T,G_D,L_y,d}(y,x) \ . \tag{7.11}$$

where we have explicitly indicated the functional dependence of the transfer matrices and eigenvalues on the variables x and y.

A. Determinants

We find

$$det(T_{T,G_D,L_y,d}) = (xy)^{L_y n_Z(G_D,L_y-1,d)} \tag{7.12}$$

for the full range $1 \le d \le L_y + 1$.

B. Traces

For the total trace, defined as

$$Tr(T_{T,G_D,L_y}) = q^{-1}\sum_{d=1}^{L_y+1} \kappa^{(d)} Tr(T_{T,G_D,L_y,d}) \tag{7.13}$$

we find

$$Tr(T_{T,G_D,L_y}) = (xy)^{L_y} \ . \tag{7.14}$$

C. Eigenvalue for $d = L_y + 1$ for $\Lambda = G_D$

For $d = L_y + 1$, the transfer matrix $T_{T,G_D,L_y,d}$ reduces to a scalar, namely

$$T_{T,G_D,L_y,L_y+1} = \lambda_{T,G_D,L_y,L_y+1} = 1 \ . \tag{7.15}$$

D. Transfer Matrix for $d = L_y$, $\Lambda = G_D$

The transfer matrix T_{T,G_D,L_y,L_y} has dimension $2L_y$. We obtain the following general formula, which we write for the Tutte polynomial, since it has a somewhat simpler form.

$$(T_{T,G_D,L_y,L_y})_{1,1} = 1 + x \tag{7.16}$$

$$(T_{T,G_D,L_y,L_y})_{j,j} = 2 + x \quad \text{for} \quad L_y \ge 2 \ \text{ and } \ 2 \le j \le L_y \tag{7.17}$$

$$(T_{T,G_D,L_y,L_y})_{j,j+1} = (T_{T,G_D,L_y,L_y})_{j+1,j} = 1 \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{7.18}$$

$$(T_{T,G_D,L_y,L_y})_{j,j} = y \quad \text{for} \quad L_y + 1 \le j \le 2L_y \tag{7.19}$$

$$(T_{T,G_D,L_y,L_y})_{j,L_y+j} = \frac{y}{y-1} \quad \text{for} \quad 1 \le j \le L_y \tag{7.20}$$

$$(T_{T,G_D,L_y,L_y})_{j+1,L_y+j} = \frac{y}{y-1} \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{7.21}$$

$$(T_{T,G_D,L_y,L_y})_{L_y+j,j} = y - 1 \quad \text{for} \quad 1 \le j \le L_y \tag{7.22}$$

$$(T_{T,G_D,L_y,L_y})_{L_y+j,j+1} = y - 1 \quad \text{for} \quad 1 \le j \le L_y - 1 \tag{7.23}$$

with all other elements equal to zero. The elements of T_{Z,G_D,L_y,L_y} are given by the relation (2.27) so that $\lambda_{Z,G_D,L_y,d} = v^{L_y}\lambda_{T,G_D,L_y,d}$.

For general x and y, T_{T,G_D,L_y,L_y} has rank equal to its dimension, $2L_y$. In the special cases $y = 0$ and $x = 0$, which yield the chromatic and flow polynomials, respectively, the rank is reduced to L_y and remains equal to L_y if both x and y are zero.

We illustrate these general formulas for the cases $L_y = 1, 2, 3$:

$$T_{T,G_D,1,1} = \begin{pmatrix} x_1 & r_d \\ v & y \end{pmatrix} \tag{7.24}$$

$$T_{T,G_D,2,2} = \begin{pmatrix} x_1 & 1 & r_d & 0 \\ 1 & x_2 & r_d & r_d \\ v & v & y & 0 \\ 0 & v & 0 & y \end{pmatrix} \tag{7.25}$$

$$T_{T,G_D,3,3} = \begin{pmatrix} x_1 & 1 & 0 & r_d & 0 & 0 \\ 1 & x_2 & 1 & r_d & r_d & 0 \\ 0 & 1 & x_2 & 0 & r_d & r_d \\ v & v & 0 & y & 0 & 0 \\ 0 & v & v & 0 & y & 0 \\ 0 & 0 & v & 0 & 0 & y \end{pmatrix} \tag{7.26}$$

Thus, in general, the upper left-hand submatrix has a main diagonal with the first entry equal to x_1 and the other entries equal to x_2. Adjacent to this main diagonal are two diagonals whose entries are 1, and the rest of the submatrix is comprised of triangular regions filled with 0's. In the upper right-hand submatrix the main diagonal and the adjacent diagonal below it have entries equal to r_d, and the other entries are zero. In the lower left-hand submatrix the main diagonal and the adjacent diagonal above it have entries equal to v, and the other entries are zero. And finally, in the right-hand lower submatrix the entries on the main diagonal are equal to y and the rest of this submatrix is made up of triangular regions of 0's. For the lowest values $L_y = 1, 2$, some of these parts, such as the triangular regions of zeros, are absent.

Using eq. (2.27), it is straightforward to obtain the T_{Z,G_D,L_y,L_y} from these T_{T,G_D,L_y,L_y} matrices. For example,

$$T_{Z,G_D,1,1} = \begin{pmatrix} q+2v & 1+v \\ v^2 & v(1+v) \end{pmatrix} \tag{7.27}$$

As corollaries of our general result for T_{T,G_D,L_y,L_y} we calculate the determinant and trace:

$$det(T_{T,G_D,L_y,L_y}) = (xy)^{L_y} \tag{7.28}$$

which is the $d = L_y$ special case of (7.12), and

$$Tr(T_{T,G_D,L_y,L_y}) = L_y(x+y) + 2L_y - 1 \ . \tag{7.29}$$

VIII. SOME ILLUSTRATIVE CALCULATIONS

A. Square-Lattice Strip, $L_y = 2$

The Potts model partition function $Z(sq, L_y \times m, BC, q, v)$ and Tutte polynomial $T(sq, L_y \times m, BC, q, v)$ (BC = boundary conditions) were calculated for the cyclic and Möbius strips of the square lattice of width $L_y = 2$ in Ref. [11]. We express the results here in terms of transfer matrices $T_{X,sq,2,d}$, $X = Z, T$, via eqs. (1.6) and (1.17):

$$T_{T,sq,2,0} = \begin{pmatrix} 1+x+x^2 & x_1 r_d \\ v & y \end{pmatrix} \tag{8.1}$$

$$T_{Z,sq,2,0} = \begin{pmatrix} q^2+3qv+3v^2 & (q+2v)(1+v) \\ v^3 & v^2(1+v) \end{pmatrix} \tag{8.2}$$

The matrices $T_{T,sq,2,1}$ and $T_{T,sq,2,2}$ have been given above. The corresponding results for the $L_y = 2$ Möbius strip follow from our general formulas also given above.

B. Triangular-Lattice Strip, $L_y = 2$

We illustrate our results for the $L_y = 2$ cyclic strip of the triangular lattice. We obtain

$$T_{T,tri,2,0} = \begin{pmatrix} (1+x)^2+y & (1+x+y)r_d \\ (y-1)(1+x+y) & y(1+y) \end{pmatrix} \tag{8.3}$$

or equivalently,

$$T_{Z,tri,2,0} = \begin{pmatrix} 5v^2+4qv+q^2+v^3 & (q+3v+v^2)(1+v) \\ v^2(q+3v+v^2) & v^2(2+v)(1+v) \end{pmatrix} \tag{8.4}$$

The matrices $T_{T,tri,2,1}$ and $T_{T,tri,2,2}$ were given above.

C. Honeycomb-lattice Strip, $L_y = 2$

For the $L_y = 2$ cyclic strip of the honeycomb lattice we calculate

$$T_{T,hc,2,0} = \begin{pmatrix} \sum_{j=0}^{4} x^j & (1+x)(1+x^2)y(y-1)^{-1} \\ y-1 & y \end{pmatrix} \tag{8.5}$$

or equivalently,

$$T_{Z,hc,2,0} = \begin{pmatrix} h_{11} & h_{12} \\ v^5 & v^4(1+v) \end{pmatrix} \tag{8.6}$$

$$h_{11} = 5v^4 + 10v^3q + 10v^2q^2 + 5vq^3 + q^4 \tag{8.7}$$

$$h_{12} = (2v+q)(2v^2+2vq+q^2)(1+v) \tag{8.8}$$

The other matrices relevant for the strip were given above.

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IX. APPENDIX

In this appendix we include tables of the numbers $\Delta n_Z(sq, L_y, d)$ and $n_Z(sq, L_y, d, \pm)$ discussed in the text.

TABLE I. Table of $\Delta n_Z(sq, L_y, d)$ for strips of the square lattice. Blank entries are zero. The last entry for each value of L_y is the total number of partitions with self-reflection symmetry.

L_y	0	1	2	3	4	5	6	7	8	9	10	$\Delta N_{Z,L_y}$
1	1	1										2
2	2	1	1									4
3	3	3	1	1								8
4	6	4	4	1	1							16
5	10	10	5	5	1	1						32
6	20	15	15	6	6	1	1					64
7	35	35	21	21	7	7	1	1				128
8	70	56	56	28	28	8	8	1	1			256
9	126	126	84	84	36	36	9	9	1	1		512
10	252	210	210	120	120	45	45	10	10	1	1	1024

TABLE II. Table of numbers $n_Z(sq, L_y, d, \pm)$ for strips of the square lattice. For each L_y value, the entries in the first and second lines are $n_Z(sq, L_y, d, +)$ and $n_Z(sq, L_y, d, -)$, respectively. Blank entries are zero. The last entry for each value of L_y is the total $N_{Z,L_y,\lambda}$.

L_y $(d,+)$ $(d,-)$	$0,+$ $0,-$	$1,+$ $1,-$	$2,+$ $2,-$	$3,+$ $3,-$	$4,+$ $4,-$	$5,+$ $5,-$	$6,+$ $6,-$	$7,+$ $7,-$	$8,+$ $8,-$	$9,+$ $9,-$	$10,+$ $10,-$	$N_{Z,L_y,\lambda}$
2	2	2	1									
		1										6
3	4	6	3	1								
	1	3	2									20
4	10	16	12	4	1							
	4	12	8	3								70
5	26	50	40	20	5	1						
	16	40	35	15	4							252
6	76	156	145	80	30	6	1					
	56	141	130	74	24	5						924
7	232	518	511	329	140	42	7	1				
	197	483	490	308	133	35	6					3432
8	750	1744	1848	1288	644	224	56	8	1			
	680	1688	1792	1260	616	216	48	7				12870
9	2494	6030	6672	5040	2772	1140	336	72	9	1		
	2368	5904	6588	4956	2736	1104	327	63	8			48620
10	8524	21100	24330	19440	11688	5352	1875	480	90	10	1	
	8272	20890	24120	19320	11568	5307	1830	470	80	9		184756

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